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Homogeneity and plane-wave limits

José Figueroa-O'Farrill and Simon Philip

School of Mathematics, University of Edinburgh
Scotland, U.K.
E-mail: J.M.Figueroa@ed.ac.uk, S.A.R.Philip@sms.ed.ac.uk

Patrick Meessen

Physics Department, Theory, CERN
Geneva, Switzerland
E-mail: Patrick.Meessen@cern.ch

Abstract: We explore the plane-wave limit of homogeneous spacetimes. For plane-wave limits along homogeneous geodesics the limit is known to be homogeneous and we exhibit the limiting metric in terms of Lie algebraic data. This simplifies many calculations and we illustrate this with several examples. We also investigate the behaviour of (reductive) homogeneous structures under the plane-wave limit.

Keywords: Penrose limit and pp-wave background, Differential and Algebraic Geometry.

In memory of Stanley E. Hobert.

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Contents

1. Introduction and conclusion

We have recently shown in [1] that M-theory backgrounds preserving more than 24 of the 32 supersymmetries are necessarily (locally) homogeneous. To date the only 24+ solutions known in the literature are actually symmetric: Minkowski spacetime, the Freund-Rubin backgrounds $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$, the Kowalski-Glikman wave [2, 3] and an Hpp-wave found by Michelson [4]. Since *all* maximally supersymmetric backgrounds are (locally) symmetric, it is not inconceivable that this might be forced by having less than maximal (say, 24?+) supersymmetry. If so, then this would allow a classification of 24?+ solutions by suitably extending results already in the literature. In the absence of such a result, however, we are forced to find other, more accessible characterisations of these backgrounds.

One property of 24+ backgrounds is that all their plane-wave limits are homogeneous. This can be established by observing that since the plane-wave limit can only enhance the preserved supersymmetry [5], any plane-wave limit of a 24+ background must itself be 24+ and hence (locally) homogeneous by [1]. This is not a trivial statement, because homogeneity is not always preserved under the plane-wave limit, as illustrated by Patricot [6], who observed that the product of a Kaigorodov spacetime with a sphere (which are homogeneous) admits nonhomogeneous plane-wave limits. (In fact, already the Kaigorodov space admits nonhomogeneous plane-wave limits.) Notice however that the plane-wave limit of a (locally) symmetric space is (locally) symmetric; hence homogeneity is preserved in these cases as well. A natural question is then: What are sufficient and necessary conditions that guarantee the preservation of homogeneity under a plane-wave limit?

Given that the data for the plane-wave limit consists of both a spacetime *and* a null geodesic on it, it should not be surprising that the homogeneity of the plane-wave limit depends on the geodesic along which the limit is taken and not just on the spacetime. Indeed, in [7] it is shown that the limit is homogeneous if it is taken along a homogeneous geodesic; that is, along the orbit of a one-parameter subgroup of isometries. This result has a clear interpretation. First of all, it is known that the isometry Lie algebra undergoes a contraction in the limit, but in general we cannot say whether the resulting algebra acts transitively or not on the resulting plane wave. The condition that the null geodesic be homogeneous then means that we force some Killing vector to remain in the translational part of the algebra. Since plane waves are generally of cohomogeneity one, this extra Killing vector guarantees homogeneity.

In view of this, it is very tempting to conjecture that the plane-wave limit of a homogeneous space is homogeneous if and only if the null geodesic is homogeneous. Even if this is not the case, the homogeneous spaces on which all null geodesics are homogeneous, are natural candidates for 24+ solutions and do merit further investigation.

In this paper we will explore the interaction between homogeneity and plane-wave limits. Section 2 is preparatory and reviews the basic technology of homogeneous manifolds, homogeneous structures, homogeneous geodesics as well as the classification of homogeneous plane waves by Blau and O'Loughlin [8]. Such plane waves come in two classes and are characterised by certain algebraic data, namely a real number and a symmetric and a skew-symmetric bilinear forms. In section 3 we show how to compute these from the

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Lie algebraic data associated to a homogeneous spacetime and a homogeneous geodesic, in effect reducing the computation of the plane-wave limit to an algebraic calculation which can be easily implemented in the computer, for example. We present two derivations of this result: one using the covariant description [9] of the plane-wave limit and one involving a different limiting procedure equivalent, as we will show, to the plane-wave limit but without the need to find neither adapted frames nor adapted coordinates. We will also study the behaviour of the homogeneous structures under the plane-wave limit. Of course, the plane-wave limit depends on the geodesic and not just on the homogeneous structure, whence the results in this section are of necessity somewhat less comprehensive. Finally, in section 4 we present some examples to illustrate our methods.

2. Homogeneity

In this section we review the basic paraphernalia of homogeneous manifolds and reductive homogeneous structures.

2.1 Basic notions

A pseudo-riemannian manifold (M, g) is *homogeneous* if some Lie group G acts transitively on M preserving the metric. Closely related to the notion of homogeneity is the notion of local homogeneity. This notion is important in the context of (super)gravitational backgrounds, since often we are only dealing with local metrics. A manifold (M, g) is *locally homogeneous* if given any two points $p, q \in M$ there are neighbourhoods $U \ni p$ and $V \ni q$ and a local isometry $f : U \to V$ such that $f(p) = q$. The crucial difference is that the isometry f need not extend to all of M. For example, the sphere is homogeneous, but the sphere without the North pole, say, is only locally homogeneous. The isometries which are defined on the sphere without the pole are those isometries which fix the pole, and these only have the parallels as orbits.

For simplicity of exposition, let us consider the case of a homogeneous manifold, so that we do have a transitive action of some group of isometries. Fixing a point $o \in M$, the smooth map

$$\phi_o : G \to M\,, \tag{2.1}$$

sending a group element $x \in G$ to its action $x \cdot o$ on the point, is surjective. The subgroup $H \subset G$ which fixes the point o is called the *isotropy subgroup* of o. The map ϕ_o induces a diffeomorphism $G/H \cong M$, which allows us to identify M with the space of right cosets of H in G in such a way that the G action of M corresponds to left multiplication on G/H. The derivative of ϕ_o at the identity $e \in G$ defines a linear map

$$d\phi_o : \mathfrak{g} \to T_o M\,, \tag{2.2}$$

where we have identified the tangent space $T_e G$ to the group at the identity with the Lie algebra $\mathfrak{g}$. Explicitly, if $X \in \mathfrak{g}$, then

$$d\phi_o(X) = \frac{d}{dt} \exp(tX) \cdot o \Big|_{t=0}\,. \tag{2.3}$$

This map is surjective with kernel the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponding to the isotropy subgroup H. In other words, we have an exact sequence:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{d\phi_o} T_oM \longrightarrow 0\,. \tag{2.4}$$

This is an exact sequence of H-modules, where H acts on $\mathfrak{h}$ and $\mathfrak{g}$ via the adjoint representation and on T_oM via the *linear isotropy representation*; that is, if $h \in H$ and $\boldsymbol{v} \in T_oM$, we define

$$h \cdot \boldsymbol{v} = \frac{d}{dt} h \cdot \gamma(t)\Big|_{t=0}\,, \tag{2.5}$$

where γ is a curve in M through o with tangent vector $\boldsymbol{v}$. This action makes T_oM isomorphic to $\mathfrak{g}/\mathfrak{h}$ as an H-module.

The metric g defines an inner product $\langle -, - \rangle$ on T_oM. Invariance of g under G is equivalent to the invariance of $\langle -, - \rangle$ under H, whence the linear isotropy representation defines a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{so}(T_oM)$. More generally, there is a bijective correspondence between H-invariant tensors on T_oM and G-invariant tensor fields on M.

We can realise the linear isotropy representation explicitly by choosing a complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and defining the action of $h \in H$ on $X \in \mathfrak{m}$ by

$$h \cdot X = (\mathrm{Ad}(h)X)_{\mathfrak{m}}\,, \tag{2.6}$$

where, here and in the following, the subscript $\mathfrak{m}$ indicates the projection onto $\mathfrak{m}$ along $\mathfrak{h}$; that is, we simply discard the $\mathfrak{h}$-component of $\mathrm{Ad}(h)X$. If $\mathfrak{m}$ is stable under $\mathrm{Ad}(H)$, so that the projection is superfluous, we say that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a *reductive split*, and the pair $(\mathfrak{g}, \mathfrak{h})$ is said to be *reductive*. This is equivalent to the splitting (in the sense of homological algebra) of the exact sequence (2.4) in the category of H-modules.

One often says that the manifold (M, g) is "reductive," but this is an abuse of language. It is important to stress that reductivity is not an intrinsic geometric property of (M, g) but of the linear isotropy representation, whence of the description of M as a coset space G/H. In fact, there are homogeneous spaces (M, g) admitting different coset descriptions G_1/H_1 and G_2/H_2, say, where one of them is reductive but not the other. The Kaigorodov space, discussed in section 4.2, is one such example: it is a left-invariant metric on a Lie group (whence trivially reductive), but as a homogeneous space of its full isometry group, it is nonreductive. Nevertheless we will say that a homogeneous pseudo-riemannian manifold (M, g) is *reductive* if there exists some G acting transitively on M via isometries, with typical isotropy H and for which the pair $(\mathfrak{g}, \mathfrak{h})$ is reductive.

If (M, g) is riemannian then $\mathrm{Ad}(H)$ is compact, whence we can always find a reductive split. This is done as follows: choose any positive-definite inner product on $\mathfrak{g}$ and average over $\mathrm{Ad}(H)$ to make it invariant. Then let $\mathfrak{m} = \mathfrak{h}^{\perp}$ be the perpendicular complement of $\mathfrak{h}$. Since $\mathfrak{h}$ is a submodule, so is $\mathfrak{m}$. If g has indefinite signature, however, reductivity is a nonempty condition. Nevertheless, it appears that all four-dimensional lorentzian homogeneous spaces [10] are indeed reductive.

2.2 Reductive homogeneous structures

For the present purposes, the importance of reductivity stems from a theorem of Ambrose and Singer [11] in the riemannian case and extended to the pseudo-riemannian case by Gadea and Oubiña [12], which provides an alternate characterisation of reductive (locally) homogeneous manifolds. As reformulated by Kostant [13], the theorem states that (M,g) is a reductive locally homogeneous pseudo-riemannian manifold if and only if there exists a metric linear connection with parallel torsion and parallel curvature. In other words, (M,g) is reductive locally homogeneous if and only if there exists a connection $\widetilde{\nabla}$ on the tangent bundle, with torsion

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] \tag{2.7}$$

and curvature

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_{[X,Y]}Z - \widetilde{\nabla}_X \widetilde{\nabla}_Y Z + \widetilde{\nabla}_Y \widetilde{\nabla}_X Z\,, \tag{2.8}$$

and such that

$$\widetilde{\nabla} g = 0 \qquad \widetilde{\nabla}\widetilde{T} = 0 \qquad \widetilde{\nabla}\widetilde{R} = 0\,. \tag{2.9}$$

These conditions define a nontrivial generalisation of the notion of a locally symmetric space, where $\widetilde{T} = 0$ and hence $\widetilde{\nabla}$ is the Levi-Cività connection. In general, the connection $\widetilde{\nabla}$ is called the *canonical connection*.

This connection can be equivalently characterised as the $\mathfrak{h}$-component of the left-invariant Maurer-Cartan one-form $\boldsymbol{\theta}$ on G, thought of as the total space of the principal H-bundle $G \to G/H$. Indeed, $\boldsymbol{\theta}$ is a $\mathfrak{g}$-valued one-form on G. Under the reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we can decompose $\boldsymbol{\theta}$ into a component along $\mathfrak{h}$ (the canonical connection) and a component along $\mathfrak{m}$ (the vielbein). Clearly, $\mathfrak{m}$ is recovered as the kernel of the canonical connection. In other words, given $\widetilde{\nabla}$ satisfying the conditions (2.9), one recovers the reductive split $\mathfrak{h} \oplus \mathfrak{m}$ by declaring $\mathfrak{m} \subset \mathfrak{g}$ to correspond to those Killing vectors which are $\widetilde{\nabla}$-parallel at the identity coset o. The geodesics of the canonical connection are given by curves of the form $\exp(tX) \cdot o$ with $X \in \mathfrak{m}$, where $o \in M$ is the identity coset.[1]

The difference between the connection $\widetilde{\nabla}$ and the Levi-Cività connection ∇ is a $(2,1)$-tensor $S : TM \to \operatorname{End} TM$, defined by

$$S_Y X := \nabla_X Y - \widetilde{\nabla}_X Y\,. \tag{2.10}$$

In fact, since both connections preserve the metric,

$$g(S_X Y, Z) = -g(Y, S_X Z)\,, \tag{2.11}$$

whence $S : TM \to \mathfrak{so}(TM)$. The space of such tensors are sections of a vector bundle $T^*M \otimes \mathfrak{so}(TM)$ associated to the bundle of orthonormal frames. Using the metric we can think of this equivalently as the sub-bundle $\mathcal{T} = T^*M \otimes \Lambda^2 T^*M \subset \otimes^3 T^*M$. This corresponds to thinking of S as a trilinear map

$$S(X,Y,Z) = g(S_X Y, Z)\,. \tag{2.12}$$

[1] For the rest of this paper, when unspecified, the word geodesic will be reserved for those of the Levi-Cività connection.

In generic dimension (here, $\dim M > 2$), the bundle $\mathcal{T}$ breaks up into the Whitney sum of three sub-bundles

$$\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \,, \tag{2.13}$$

each one corresponding to an irreducible representation of the orthogonal group. In terms of Young tableaux, this decomposition is given by

$$\begin{gathered} T^* \otimes \Lambda^2 T^* = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \\ \ydiagram{1} \otimes \ydiagram{1,1} = \ydiagram{1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1} \,. \end{gathered} \tag{2.14}$$

More explicitly, the bundles $\mathcal{T}_i$ can be described as follows:

1. Sections of $\mathcal{T}_1$ correspond to sections S of $\mathcal{T}$ such that

$$S(X, Y, Z) = g(X, Y)\alpha(Z) - g(X, Z)\alpha(Y) \tag{2.15}$$

   for some one-form α, whence $\mathcal{T}_1 \cong T^*M$.

2. Sections of $\mathcal{T}_2$ correspond to sections S of $\mathcal{T}$ which satisfy

$$S(X, Y, Z) + S(Y, Z, X) + S(Z, X, Y) = 0 \,, \tag{2.16}$$

   and are in the kernel of the map $C : \otimes^3 T^*M \to T^*M$ defined by contracting with the (inverse) metric on the first two indices:

$$C(S)(X) = \sum_{a,b} g^{ab} S(\boldsymbol{e}_a, \boldsymbol{e}_b, X) \,, \tag{2.17}$$

   where $\boldsymbol{e}_a$ is a pseudo-orthonormal frame and g^{ab} is the inverse of $g_{ab} = g(\boldsymbol{e}_a, \boldsymbol{e}_b)$.

3. Sections of $\mathcal{T}_3$ correspond to sections of $\mathcal{T}$ which are totally skew-symmetric, whence $\mathcal{T}_3 \cong \Lambda^3 T^*M$.

It is easy to write down the explicit expressions for each of the components of S. We will write $S_{abc} = S(\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c)$ relative to a pseudo-orthonormal frame. Then

$$S_{abc} = S^{\ydiagram{1}}_{abc} + S^{\ydiagram{2,1}}_{abc} + S^{\ydiagram{1,1,1}}_{abc} \,, \tag{2.18}$$

where

$$\begin{aligned} S^{\ydiagram{1}}_{abc} &= g_{ab}\xi_c - g_{ac}\xi_b \\ S^{\ydiagram{1,1,1}}_{abc} &= \frac{1}{3}\left(S_{abc} + S_{bca} + S_{cab}\right) \\ S^{\ydiagram{2,1}}_{abc} &= S_{abc} - S^{\ydiagram{1}}_{abc} - S^{\ydiagram{1,1,1}}_{abc} \,, \end{aligned} \tag{2.19}$$

where

$$\xi_c = \frac{1}{n-1} g^{ab} S_{abc} \,, \tag{2.20}$$

with $n = \dim M$.

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Given a reductive locally homogeneous space (M,g), the Ambrose-Singer theorem guarantees the existence of a tensor S, called a *(reductive) homogeneous structure*. Following Tricerri and Vanhecke [14], we can distinguish eight ($= 2^3$) types of homogeneous structures, depending on whether S does or does not have a component in each of the three irreducible components $\mathcal{T}_i$. This condition can be probed at any one point $o \in M$ because S is parallel with respect to a metric connection $\widetilde{\nabla}$, hence its type under the orthogonal group does not change under parallel transport with respect to $\widetilde{\nabla}$.

The eight possible homogeneous structures are the following:[2]

1. $S = 0$: the locally symmetric spaces;

2. $S \in \mathcal{T}_1$: here there exists a vector ξ such that S takes the form

$$S_X Y = g(X,Y)\xi - g(Y,\xi)X\,. \tag{2.21}$$

   In riemannian signature, Tricerri and Vanhecke [14] proved that (M,g) has constant negative curvature, whence it is locally isometric to hyperbolic space; that is, locally symmetric. In lorentzian signature, we must distinguish between two cases: according to whether the norm of ξ is zero or nonzero. In the latter case, Gadea and Oubiña [12] proved that (M,g) is locally isometric to anti-de Sitter space, whereas if ξ is null, then Montesinos Amilibia [15] showed that (M,g) is a singular homogeneous plane-wave [8]:

$$g = 2dudv + A(\boldsymbol{x},\boldsymbol{x})\frac{du^2}{u^2} + |d\boldsymbol{x}|^2\,, \tag{2.22}$$

   with A a constant bilinear form;

3. $S \in \mathcal{T}_2$;

4. $S \in \mathcal{T}_3$: these are the so-called naturally reductive homogeneous spaces. They can be alternatively characterised as those reductive homogeneous manifolds for which $\exp(tX)\cdot o$ is a geodesic through o for every nonzero $X \in \mathfrak{m}$;

5. $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$: in this case the tensor S satisfies equation (2.16);

6. $S \in \mathcal{T}_2 \oplus \mathcal{T}_3$: in this case the tensor S is in the kernel of the contraction map C defined in (2.17);

7. $S \in \mathcal{T}_1 \oplus \mathcal{T}_3$: in this case, the tensor S satisfies

$$S(X,Y,Z) + S(Y,X,Z) = 2g(X,Y)\alpha(Z) - g(X,Z)\alpha(Y) - g(Z,Y)\alpha(X)\,, \tag{2.23}$$

   for some one-form α. It is shown in [16] that if α has non-zero norm, then the underlying geometry is once again that of a symmetric space, whereas if α is null then it is a generic singular homogeneous plane-wave [8]

$$g = 2dudv + \left[A(e^{-uF}\boldsymbol{x}, e^{-uF}\boldsymbol{x}) + 2v\right]du^2 + |d\boldsymbol{x}|^2\,, \tag{2.24}$$

[2] We abuse notation slightly and identify the bundles $\mathcal{T}_i$ with their sheaves of sections, whence $S \in \mathcal{T}_i$ means that S is a section of $\mathcal{T}_i$, etc.

where A is once again a constant bilinear form and F is a skew-symmetric matrix; and, finally,

8. S generic.

It must be stressed that a given homogeneous space can admit more than one homogeneous structure, as indeed can be seen from the examples in section 4 or from the characterisation of the non-degenerate $\mathcal{T}_1$ class. We can understand this as follows. There is a one-to-one correspondence between homogeneous structures S and reductive splits $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In principle, different choices of $\mathfrak{h}$ and $\mathfrak{m}$ give rise to different homogeneous structures. Indeed, given $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the homogeneous structure S at the identity coset o is given by

$$S(X, Y, Z) = g\left(\nabla_Y X\big|_o, Z\right), \tag{2.25}$$

where X, Y, Z are Killing vectors in $\mathfrak{m}$.

Now suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive split with maximal $\mathfrak{g}$, and let $\mathfrak{g}' \subseteq \mathfrak{g}$ be a subalgebra such that the restriction of the map $\mathfrak{g} \to T_o M$ to $\mathfrak{g}'$ is still surjective. Let $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ and let $\mathfrak{m}' = \mathfrak{g}' \cap \mathfrak{m}$. Surjectivity implies that $\mathfrak{m}' = \mathfrak{m}$, whence $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}$ is still a reductive split. This split can be deformed as follows. We pick a subspace $\mathfrak{m}' \subset \mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ is still a reductive split. This means that $\mathfrak{m}'$ is the graph of an $\mathfrak{h}'$-equivariant linear map $\varphi : \mathfrak{m} \to \mathfrak{h}'$; that is,

$$\mathfrak{m}' = \{\varphi(X) + X \mid X \in \mathfrak{m}\}. \tag{2.26}$$

As $\mathfrak{h}'$ grows, there are more linear maps $\mathfrak{m} \to \mathfrak{h}'$, but also the $\mathfrak{h}'$-equivariance condition becomes stronger. It is therefore not inconceivable that we should obtain nontrivial φ's by restricting to subalgebras as just described. We will see an example of this in section 4. Observe, by the way, that conjugate subalgebras yield isomorphic homogeneous structures.

Given a homogeneous structure S, the Lie bracket restricted to the subspace $\mathfrak{m}$ of the isometry algebra is given by the formula

$$[X, Y] = S_X Y - S_Y X + \widetilde{R}(X, Y), \tag{2.27}$$

where $X, Y \in \mathfrak{m}$ and S and $\widetilde{R}$ are evaluated at the point o. This defines the subspace $\mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$, from which we may define the full reductive split $\mathfrak{m} \oplus \mathfrak{h}$ to be the algebraic closure of this subspace under the Lie bracket (2.27) together with

$$[A, X] = A(X) \quad \text{and} \quad [A, B] = AB - BA, \tag{2.28}$$

where $X \in \mathfrak{m}$ and $A, B \in \mathrm{End}(\mathfrak{m})$. Notice that not all elements of $\mathfrak{h}$ need appear in $\widetilde{R}$, in fact the holonomy algebra $\mathrm{hol}(\widetilde{\nabla})$ must be an ideal of $\mathfrak{h}$.

2.3 Calculating with homogeneous spaces

We now collect some useful formulae for calculating the Riemann tensor of a homogeneous space in terms of Lie algebraic data. For more details one can consult, for example, the book [17].

Let X, Y, Z be Killing vectors on $M = G/H$. The Koszul formula for the Levi-Cività connection reads

$$g(\nabla_X Y, Z) = \frac{1}{2} g([X,Y],Z) + \frac{1}{2} g([X,Z],Y) + \frac{1}{2} g(X,[Y,Z])\,. \tag{2.29}$$

At the identity coset $o \in M$ and assuming that X, Y, Z are Killing vectors in $\mathfrak{m}$, then

$$\nabla_X Y\big|_o = -\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y)\,, \tag{2.30}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is a symmetric tensor given by[3]

$$\langle U(X,Y), Z\rangle = \frac{1}{2}\langle [Z,X]_{\mathfrak{m}}, Y\rangle + \frac{1}{2}\langle [Z,Y]_{\mathfrak{m}}, X\rangle\,, \tag{2.31}$$

for all $Z \in \mathfrak{m}$. It should be remarked that (2.30) is only valid at $o \in M$, since $\nabla_X Y$ is not generally a Killing vector. Of course, since ∇ is G-invariant, then one can determine $\nabla_X Y\big|_p$ at any other point by acting with any isometry relating o and p.

The formula (2.25) for the corresponding homogeneous structure (at o) can now be written explicitly:

$$S(X,Y,Z) = \frac{1}{2}\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \frac{1}{2}\langle [Z,X]_{\mathfrak{m}}, Y\rangle + \frac{1}{2}\langle [Z,Y]_{\mathfrak{m}}, X\rangle\ , \tag{2.32}$$

for $X, Y, Z \in \mathfrak{m}$.

The U-tensor is not generally invariant under the linear isotropy representation; indeed, for all $Z \in \mathfrak{h}$,

$$(Z \cdot U)(X,Y) = [[Z,X]_{\mathfrak{h}}, Y]_{\mathfrak{m}} + [[Z,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}\,; \tag{2.33}$$

although it clearly does when G/H is reductive. The vanishing of the U-tensor characterises the *naturally reductive* homogeneous structures.

The Riemann curvature tensor is G-invariant and it can be computed at o. One obtains, for X, Y, Z, W vectors in $\mathfrak{m}$, the curvature tensor at o is given by

$$\begin{aligned}
R(X,Y,Z,W) = {} & \langle U(X,W), U(Y,Z)\rangle - \langle U(X,Z), U(Y,W)\rangle + \\
& + \frac{1}{12}\langle [X,[Y,Z]]_{\mathfrak{m}}, W\rangle - \frac{1}{12}\langle [X,[Y,W]]_{\mathfrak{m}}, Z\rangle - \\
& - \frac{1}{6}\langle [X,[Z,W]]_{\mathfrak{m}}, Y\rangle - \frac{1}{12}\langle [Y,[X,Z]]_{\mathfrak{m}}, W\rangle + \\
& + \frac{1}{12}\langle [Y,[X,W]]_{\mathfrak{m}}, Z\rangle + \frac{1}{6}\langle [Y,[Z,W]]_{\mathfrak{m}}, X\rangle - \\
& - \frac{1}{6}\langle [Z,[X,Y]]_{\mathfrak{m}}, W\rangle - \frac{1}{12}\langle [Z,[X,W]]_{\mathfrak{m}}, Y\rangle + \\
& + \frac{1}{12}\langle [Z,[Y,W]]_{\mathfrak{m}}, X\rangle + \frac{1}{6}\langle [W,[X,Y]]_{\mathfrak{m}}, Z\rangle + \\
& + \frac{1}{12}\langle [W,[X,Z]]_{\mathfrak{m}}, Y\rangle - \frac{1}{12}\langle [W,[Y,Z]]_{\mathfrak{m}}, X\rangle -
\end{aligned}$$

[3]The apparent difference in sign between equation (2.29) and equations (2.30) and (2.31) stems from the fact that Killing vectors on G/H generate left translations on G, whence they are right-invariant. Thus the map $\mathfrak{g} \to$ Killing vector fields is an anti-homomorphism.

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$$-\frac{1}{2}\langle[X,Y]_{\mathfrak{m}},[Z,W]_{\mathfrak{m}}\rangle-\frac{1}{4}\langle[X,Z]_{\mathfrak{m}},[Y,W]_{\mathfrak{m}}\rangle+$$
$$+\frac{1}{4}\langle[X,W]_{\mathfrak{m}},[Y,Z]_{\mathfrak{m}}\rangle\ , \tag{2.34}$$

which can be obtained by polarisation from the simpler expression for $K(X,Y) := \langle R(X,Y)X,Y\rangle$, which is also easier to derive. Indeed, and for completeness, one has

$$\begin{aligned}6R(X,Y,Z,W) = {} & K(X+Z,Y+W)-K(Y+Z,X+W)-\\ & -K(Y+W,X)+K(Y+Z,X)-K(X+Z,Y)+K(X+W,Y)-\\ & -K(Y+W,Z)+K(X+W,Z)-K(X+Z,W)+K(Y+Z,W)+\\ & +K(X,W)-K(X,W)-K(Y,W)+K(Y,Z)-K(X,Z)\,,\end{aligned} \tag{2.35}$$

where

$$\begin{aligned}K(X,Y) = {} & -\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2-\frac{1}{2}\langle[X,[X,Y]]_{\mathfrak{m}},Y\rangle-\frac{1}{2}\langle[Y,[Y,X]]_{\mathfrak{m}},X\rangle+\\ & +|U(X,Y)|^2-\langle U(X,X),U(Y,Y)\rangle\end{aligned} \tag{2.36}$$

and where $|-|^2$ is the (indefinite) norm associated to $\langle -,-\rangle$.

2.4 Geodesics in homogeneous spaces

We shall be interested in geodesics in G/H which are themselves orbits of one-parameter subgroups of G. By homogeneity we can assume that the geodesics pass through our base point o. Such *homogeneous* geodesics can always be reparameterised so that they are given by

$$\gamma(t)=\exp(tX)\cdot o\,, \tag{2.37}$$

for some *geodetic vector* $X\in\mathfrak{g}$. The condition of $X\in\mathfrak{g}$ being geodetic is that the curve traced by γ above be a geodesic. If γ' has non-zero norm, this is equivalent to the self-parallel condition $\nabla_{\gamma'}\gamma'=0$, but for null geodesics, one can relax this condition to $\nabla_{\gamma'}\gamma'=c(\gamma)\,\gamma'$. It follows from the Koszul formula (2.29) that $X\in\mathfrak{g}$ is geodetic if and only if

$$\langle[X,Z]_{\mathfrak{m}},X_{\mathfrak{m}}\rangle=c\,\langle X_{\mathfrak{m}},Z\rangle\ , \tag{2.38}$$

for all $Z\in\mathfrak{m}$, and where c is some constant.

If X in equation (2.37) belongs to $\mathfrak{m}$, then we say that the geodesic is *canonically homogeneous*, since then γ is also a geodesic for the canonical connection.

There are some spaces for which all geodesics are homogeneous. This is the case, for example, for the naturally reductive spaces in which the U tensor defined in (2.31) vanishes. In fact, for such spaces every geodesic is canonically homogeneous. More generally, a space for which all geodesics are homogeneous is called a *geodesic orbit* space or *g.o.* space, for short. In section 4.4 we will discuss an example of a g.o. space, a six-dimensional lorentzian manifold of the type first considered by Kaplan (see, for example, [18]). In a g.o. space, given any nonzero $X\in\mathfrak{m}$, there is an element $\phi(X)\in\mathfrak{h}$ such that $X+\phi(X)\in\mathfrak{g}$ is geodetic; that is,

$$\langle[\phi(X),X]_{\mathfrak{m}}+c(X)X,Z\rangle=\langle[X,Z]_{\mathfrak{m}},X\rangle\,, \tag{2.39}$$

for all $Z\in\mathfrak{m}$.

2.5 Homogeneous plane waves

Homogeneous plane waves have been recently classified. In [8] it is proved that they fall in two classes: one consisting of regular homogeneous plane waves and one consisting of singular homogeneous plane waves.

In Brinkmann coordinates the regular plane-wave metric takes the form

$$2dudv + A(e^{-uF}\boldsymbol{x}, e^{-uF}\boldsymbol{x})du^2 + |d\boldsymbol{x}|^2\,, \tag{2.40}$$

where A is a constant symmetric bilinear form and F is a constant skew-symmetric matrix. When $[F, A] = 0$, F drops out of the metric and the resulting metric is symmetric. Otherwise, the space is not locally symmetric, but is a naturally reductive space as evidenced by the existence of a $\mathcal{T}_3$ structure

$$S = \frac{1}{2}F_{ij}du \wedge dx^i \wedge dx^j\,. \tag{2.41}$$

Furthermore, it can be shown that regular homogeneous plane waves do not admit homogeneous structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$.

The metric for the singular homogeneous plane waves in Brinkmann coordinates reads

$$2e^z dzdv + A(e^{-zF}\boldsymbol{x}, e^{-zF}\boldsymbol{x})dz^2 + |d\boldsymbol{x}|^2\,. \tag{2.42}$$

We can change coordinates and rewrite this metric as

$$2dudv + A(e^{-(\log u)F}\boldsymbol{x}, e^{-(\log u)F}\boldsymbol{x})\frac{du^2}{u^2} + |d\boldsymbol{x}|^2\,, \tag{2.43}$$

which has manifestly a pp-wave singularity at $u = 0$, whence it is incomplete. In general it admits a $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure given by

$$S_{uuv} = \frac{1}{u} \qquad S_{uij} = \frac{1}{u}F_{ij} \qquad S_{iuj} = \frac{1}{u}\left[\delta_{ij} - F_{ij}\right], \tag{2.44}$$

and it admits an $\mathcal{T}_1$ structure when $[F, A] = 0$, which is just the same as taking $F = 0$. In fact, as was said in section 2.2, the singular homogeneous plane waves with $F = 0$ are the only spacetimes admitting a degenerate $\mathcal{T}_1$ structure.

Let us remark that if one tries to repeat the analysis of [8] by means of the Ambrose-Singer equations based on the metric

$$2dudv + A_{ij}(u)x^i x^j du^2 + |d\boldsymbol{x}|^2\,, \tag{2.45}$$

one finds that the only solutions for S_{uuv} are either zero or $1/(u + u_0)$, signalling a regular or a singular plane wave, respectively.

3. Homogeneous plane-wave limits

In this section we describe in very concrete terms the plane-wave limit of a homogeneous spacetime along a homogeneous null geodesic. We give two derivations of algebraic formulae — equations (3.53) and (3.55) — for the limiting metric in terms of the initial data describing the homogeneous spacetime and the geodesic in question. One derivation uses the covariant characterisation of the plane-wave limit given in [9], whereas the other involves a limiting procedure different yet equivalent, as we will show, to the plane-wave limit, and which does away with the need to to find neither an adapted frame nor adapted coordinates. We start, though, with a brief review of the plane-wave limit itself.

3.1 Plane-wave limits

Let γ be a null geodesic in a spacetime (M, g). Then according to Penrose [19] a coordinate system (u, v, y^i) can be found in a neighbourhood of γ without conjugate points, such that the metric takes the form

$$g = dudv + \alpha dv^2 + \sum_i \beta_i dy^i dv + \sum_{i,j} C_{ij} dy^i dy^j \,, \tag{3.1}$$

where α, β_i and C_{ij} are smooth functions and we have $\gamma' = \partial_u$. Following Penrose, we now rescale the coordinates by a factor $\Omega \in \mathbb{R}$

$$v \mapsto \Omega^2 v \,, \quad y^i \mapsto \Omega y^i \,, \quad u \mapsto u \,, \tag{3.2}$$

and denote the metric we obtain from this redefinition by g_Ω. Then the limit as $\Omega \to 0$ of $\Omega^{-2} g_\Omega$ is well defined

$$\begin{aligned} \overline{g} &= \lim_{\Omega \to 0} \left(dudv + \Omega^2 \alpha dv^2 + \Omega \sum_i \beta_i dy^i dv + \sum_{i,j} C_{ij} dy^i dy^j \right) \\ &= dudv + \sum_{i,j} C_{ij}(u, 0, 0) dy^i dy^j \end{aligned} \tag{3.3}$$

and is called the *Penrose* or *plane-wave limit along* γ. This limiting procedure has been extended by Güven [20] to supergravity theories with additional fields and provides a method for obtaining new supergravity solutions from old.

As was shown in [5], there are geometric properties of the spacetime which are preserved under plane-wave limits. Following Geroch [21] one calls these properties *hereditary*, since the limiting spacetime inherits them from the parent spacetime. Such hereditary properties include the Killing spinors and Killing vectors; although it is not uncommon that the limiting spacetime is more (super)symmetric than the parent spacetime.

Another example of a hereditary property [21, 5] is that of being locally symmetric; that is, if the Riemann curvature tensor is parallel before the limit, it is parallel after the limit. Furthermore, since every locally symmetric plane-wave, or Cahen-Wallach space, is geodesically complete, we can strengthen this result and claim that the plane-wave limit of a locally symmetric space is symmetric, after suitable completion. Since a locally symmetric space is locally homogeneous, we see that homogeneity can be hereditary under the plane-wave limit even though this is not the case in general, as evidenced by the Kaigorodov space [6]. This prompts the following question: under what extra conditions is (local) homogeneity preserved? The remainder of this section is devoted to exploring this question.

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If γ is a homogeneous geodesic then it was shown in [7] that the plane-wave limit along γ is locally homogeneous as defined in section 2.1. This was done by examining the Killing transport along γ and showing that it has a well-defined plane-wave limit which generates a Killing vector field for the plane-wave limit metric. As the generic plane wave is of cohomogeneity one, this extra Killing vector gives the result.

The above gives a sufficient condition on a null geodesic, in a generic spacetime, for the plane-wave limit along it to be homogeneous. It is however not a necessary condition as the following example shows. Consider the metric

$$2dudv + udv^2 + \sqrt{u}\sum_i (dx^i)^2 \,. \tag{3.4}$$

This is an incomplete and nonhomogeneous metric, with no Killing vector in the ∂_u direction. Therefore the null geodesic given by ∂_u is not homogeneous. The plane-wave limit along this geodesic,

$$2dudv + \sqrt{u}\sum_i (dx^i)^2 \,, \tag{3.5}$$

is however a singular homogeneous plane wave [22].

The above raises the question about the existence of homogeneous (null) geodesics in reductive homogeneous spaces. It is a theorem by Kowalski and Szenthe [23], suitably extended to the lorentzian case [7], that in a reductive lorentzian homogeneous manifold there exists at least one (not necessarily null) homogeneous geodesic through every point. On the other hand, on a g.o. spacetime every geodesic is homogeneous, whence homogeneity is hereditary for these spacetimes.

The existence of particular classes of homogeneous structures on a reductive homogeneous spacetime can indicate the existence of null homogeneous geodesics. Indeed, if S is a section of $\mathcal{T}_1 \oplus \mathcal{T}_3$, then for a null geodesic γ of the $\widetilde{\nabla}$ connection we have

$$0 = \widetilde{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' - g(\gamma',\gamma')\xi + g(\gamma',\xi)\gamma' = \nabla_{\gamma'}\gamma' + g(\gamma',\xi)\gamma' \,. \tag{3.6}$$

Now if we reparameterise $\gamma(\tau)$ to $\overline{\gamma}(s)$, such that $\gamma' = \partial_\tau = f(s)\partial_s = f(s)\overline{\gamma}'$, we find that

$$0 = f^2\nabla_{\overline{\gamma}'}\overline{\gamma}' + f\left(\nabla_{\overline{\gamma}'} f\right)\overline{\gamma}' + f^2 g(\overline{\gamma}'\ ,\xi)\overline{\gamma}' \,. \tag{3.7}$$

So that a solution to

$$\frac{\partial f}{\partial s} + g(\overline{\gamma}'\ ,\xi) f = 0 \tag{3.8}$$

maps a null geodesic of $\widetilde{\nabla}$ to a null geodesic of ∇. Conversely, given a null geodesic for ∇ we can perform the inverse transformation and obtain a null geodesic for $\widetilde{\nabla}$. Thus, every null geodesic in a spacetime with a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is canonically homogeneous and hence the plane-wave limit of the spacetime is always a homogeneous plane wave admitting a homogeneous structure contained in $\mathcal{T}_1 \oplus \mathcal{T}_3$. In fact, it has been shown in [16] that a reductive homogeneous space admitting a structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is either a locally symmetric space (and hence naturally reductive), a singular homogeneous plane-wave or a more general naturally reductive space, depending on whether the vector in the $\mathcal{T}_1$-component is nondegenerate, null or zero, respectively.

In section 3.4 we will show, by considering various examples, that the existence of one of the other classes of homogeneous structures on a spacetime says little about the existence of homogeneous geodesics.

3.2 The covariant method

Let g be a lorentzian metric and γ a null geodesic of g. Consider g to be written in a twist-free coordinate system (3.1) and let $(\partial_u, \partial_v, \partial_i)$ denote the dual frame to (du, dv, dy^i).

In [9], the following covariant formulation of the plane-wave limit is given. We say that a local frame (E_+, E_-, E_a) is *adapted to a null geodesic* γ, if the following conditions are satisfied:

1. E_+ is a geodesic vector field such that $E_+|_\gamma$ is proportional to $\partial_u|_\gamma$, where u is the parameter along γ;
2. $\nabla_u E_- = \nabla_u E_a = 0$ along γ; and
3. the metric takes the form
$$g = 2\theta^+\theta^- + \sum_a \theta^a\theta^a \tag{3.9}$$
where the θ's are the dual coframe.

Let (E_+, E_-, E_a) be such an adapted frame. We can write E_a in the form

$$E_a = E_a^i\partial_i + E_a^u\partial_u + E_a^v\partial_v\,. \tag{3.10}$$

By taking its inner product with E_+ and with E_b we see that, restricted to the geodesic γ, we have

$$E_a^v = 0 \tag{3.11}$$

and

$$E_{ai}E_b^i = C_{ij}E_a^jE_b^i = \delta_{ab}\,. \tag{3.12}$$

Calculating the covariant derivative of E_a we have

$$(E_a^i)' + E_a^j\Gamma^i_{ju} = 0 \tag{3.13}$$

and the dual equation

$$(E_{ai})' - E_{aj}\Gamma^j_{iu} = 0\,. \tag{3.14}$$

Thus

$$(E_{ai})'E_b^i = -E_{ai}(E_b^i)' = E_{ai}E_b^j\Gamma^i_{ju} = E_a^iE_{bj}\Gamma^i_{ju} = E_a^i(E_{bj})'\,. \tag{3.15}$$

Now consider the plane-wave limit $\overline{g}$ of the metric g. A frame E_M satisfying equation (3.15) defines a change of coordinates from the Rosen coordinate description of $\overline{g}$ to a Brinkmann coordinate description

$$2dx^+dx^- + A_{ij}(x^+)x^ix^j(dx^+)^2 + \sum_i (dx^i)^2\,, \tag{3.16}$$

where

$$A_{ab}(x^+) = -R(E_+, E_a, E_+, E_b)|_\gamma = -R(E_+, \partial_i, E_+, \partial_j)|_\gamma E^i_a|_\gamma E^j_b|_\gamma\,. \qquad (3.17)$$

This covariant description of the plane-wave limit illustrates that the limit is really an invariant of the null geodesic and not just a remnant of a special coordinate system. However, it is not much easier to apply than the usual plane-wave limit as finding a parallel frame can be difficult. On the other hand, on reductive spaces it is a fruitful approach.

Indeed, suppose that now (M, g) is a reductive homogeneous space with a homogeneous structure S. Let M be locally isomorphic to the quotient G/H and let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ be the reductive split of the Lie algebra of G associated to S. Let $U \in \mathfrak{g}$ be the geodetic vector that determines γ as homogeneous. Let $V \in \mathfrak{m}$ be the dual null vector and complete to a basis with orthonormal elements $Y_i \in \mathfrak{m}$. The classification of homogeneous plane waves [8] states that the plane-wave limit in Brinkmann coordinates will be of the form:

$$A(x^+) = e^{x^+F} A_0 e^{-x^+F} \qquad \text{or} \qquad A(x^+) = e^{\log(cx^+)F} A_0 e^{-\log(cx^+)F}/(cx^+)^2\,, \qquad (3.18)$$

where A_0 is a nondegenerate symmetric bilinear form, F is a skew-symmetric bilinear form and $c \neq 0$ is the constant in (2.38). The first case corresponds to the non-singular plane-waves and the second to the singular waves. We shall take the origin o for the non-singular waves to be the point $(0, 0, 0)$, while for the singular waves we take $(1/c, 0, 0)$.

We will now use the above covariant description and the algebraic description of the curvature tensor on such a background to write down an algebraic formula for both A_0 and F.

Let E_M be an adapted frame to the geodesic γ which when restricted to o corresponds to the basis (U, V, Y_i). For a non-singular homogeneous plane-wave limit $\overline{g}$ we have

$$\exp(x^+[F, -]) \cdot A_0 = A_{ab}(x^+) = -R(E_+, E_a, E_+, E_b)|_\gamma\,. \qquad (3.19)$$

Thus, evaluating at o,

$$\begin{aligned}(A_0)_{ab} &= -R(E_+, E_a, E_+, E_b)|_0 \\ &= -R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b)\,, \end{aligned} \qquad (3.20)$$

where $U_\mathfrak{m}$ is the projection to $\mathfrak{m}$ of $U \in \mathfrak{g}$ and $Y_a = E_a(0) \in \mathfrak{m}$. Similarly, we find that (3.20) holds for the singular plane-waves.

Now, if we differentiate the left hand side of equation (3.19) and evaluate at o we obtain

$$\frac{\partial}{\partial x^+}\left(A_{ab}(x^+)\right)\Big|_o = -2cA_0 + [F, A_0]\,. \qquad (3.21)$$

Differentiating the right hand side,

$$\begin{aligned}\frac{\partial}{\partial x^+}\left(A_{ab}(x^+)\right)|_o &= -\frac{\partial}{\partial x^+}R(E_+, E_a, E_+, E_b)|_o\,, \\ &= -\nabla_U(R(E_+, E_a, E_+, E_b)|_\gamma)\,, \\ &= -\left(\nabla_U R(E_+, E_a, E_+, E_b)\right)|_\gamma\,, \\ &= -\left(\nabla_U R\right)(E_+, E_a, E_+, E_b)|_\gamma\,, \end{aligned} \qquad (3.22)$$

where we have used the fact that U is a vector field tangent to γ and that the frame E_M is parallel with respect to U.

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The object ∇R is tensorial, that is

$$(\nabla R)(\cdot,\ldots,fX,\ldots,\cdot) = f(\nabla R)(\cdot,\ldots,X,\ldots,\cdot)\,, \tag{3.23}$$

for any $f \in C^\infty(M)$, whence, by passing the restriction to 0 through the curvature, we have

$$\frac{\partial}{\partial x^+}\left(A_{ab}(x^+)\right)|_0 = -\left(\nabla_{U_\mathfrak{m}} R\right)\left(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b\right). \tag{3.24}$$

As $U_\mathfrak{m}$ is a Killing vector [24]

$$(\nabla_{U_\mathfrak{m}} - S_{U_\mathfrak{m}}\cdot)R = \mathfrak{L}_{U_\mathfrak{m}} R = 0\,. \tag{3.25}$$

Hence we can replace the differential action of the covariant derivative with the algebraic action of the linear map $S_{U_\mathfrak{m}}$,

$$\begin{aligned}\frac{\partial}{\partial x^+}\left(A_{ab}(x^+)\right)|_0 &= -\left(S_{U_\mathfrak{m}}\cdot R\right)\left(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b\right),\\ &= R(S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_a, U_\mathfrak{m}, Y_b) +\\ &\quad + R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_b)\,,\end{aligned} \tag{3.26}$$

where we have used that the action of $S_{U_\mathfrak{m}}$ annihilates functions. Therefore we obtain the formula

$$\begin{aligned}-2c(A_0)_{ab} + [F, A_0]_{ab} &= R(S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_a, U_\mathfrak{m}, Y_b) +\\ &\quad + R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_b)\,.\end{aligned} \tag{3.27}$$

Similarly, differentiating a second time and evaluating at zero, we find that $(6c^2 A_0 - 3c[F, A_0] + [F, [F, A_0]])_{ab}$ is given by

$$\begin{aligned}&R(S_{U_\mathfrak{m}}S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{U_\mathfrak{m}}S_{U_\mathfrak{m}}Y_a, U_\mathfrak{m}, Y_b) +\\ &+R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}}S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}}S_{U_\mathfrak{m}}Y_b) +\\ &+2R(S_{U_\mathfrak{m}}U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_a, U_\mathfrak{m}, Y_b) + 2R(S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_b) +\\ &+2R(S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_b) + 2R(U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_a, S_{U_\mathfrak{m}}U_\mathfrak{m}, Y_b) +\\ &+2R(U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_a, U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_b) + 2R(U_\mathfrak{m}, Y_a, S_{U_\mathfrak{m}}U_\mathfrak{m}, S_{U_\mathfrak{m}}Y_b) +\\ &+R(S_{S_{U_\mathfrak{m}}U_\mathfrak{m}}U_\mathfrak{m}, Y_a, U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, S_{S_{U_\mathfrak{m}}U_\mathfrak{m}}Y_a, U_\mathfrak{m}, Y_b) +\\ &+R(U_\mathfrak{m}, Y_a, S_{S_{U_\mathfrak{m}}U_\mathfrak{m}}U_\mathfrak{m}, Y_b) + R(U_\mathfrak{m}, Y_a, U_\mathfrak{m}, S_{S_{U_\mathfrak{m}}U_\mathfrak{m}}Y_b)\,.\end{aligned} \tag{3.28}$$

Similar expressions can be obtained for higher order brackets between F and A_0. By calculating enough terms of the form $[F,\ldots,[F,A_0]]$, one can solve for the skew-symmetric matrix F, but in fact, it is not difficult to write down a general solution.

First we note that since U is geodetic, we have

$$S_{U_\mathfrak{m}}U_\mathfrak{m} + S_{U_\mathfrak{h}}U_\mathfrak{m} = S_U U_\mathfrak{m} = -c\, U_\mathfrak{m}\,, \tag{3.29}$$

where we are extending[4] the definition (2.32) of S to the whole of $\mathfrak{g}$ by $S_Y X = \nabla_X Y$. Together with invariance of the curvature, this allows one to manipulate (3.27)

$$\begin{aligned}[F, A_0]_{ab} &= R(U_{\mathfrak{m}}, (S_{U_{\mathfrak{m}}} + S_{U_{\mathfrak{h}}})Y_a, U_{\mathfrak{m}}, Y_b) + R(U_{\mathfrak{m}}, Y_a, U_{\mathfrak{m}}, (S_{U_{\mathfrak{m}}} + S_{U_{\mathfrak{h}}})Y_b) \\ &= \langle R(Y_b, U_{\mathfrak{m}})U_{\mathfrak{m}}, S_U Y_a\rangle + \langle R(U_{\mathfrak{m}}, Y_a)U_{\mathfrak{m}}, S_U Y_b\rangle \,. \end{aligned} \tag{3.30}$$

Recall that $(A_0)_{ab} = -R(U_{\mathfrak{m}}, Y_a, U_{\mathfrak{m}}, Y_b)$, therefore, we can take F to be

$$F_{ab} = -\langle S_U(Y_a), Y_b\rangle = S(U, Y_b, Y_a) \tag{3.31}$$

where we have used that

$$\langle S_U Y_a, U_{\mathfrak{m}}\rangle = -c\langle Y_a, U_{\mathfrak{m}}\rangle = 0 \tag{3.32}$$

and thus

$$\langle S_U Y_a, U_{\mathfrak{m}}\rangle\langle V, R(U_{\mathfrak{m}}, Y_a)U_{\mathfrak{m}}\rangle = \langle S_U Y_a, V\rangle\langle U_{\mathfrak{m}}, R(U_{\mathfrak{m}}, Y_a)U_{\mathfrak{m}}\rangle = 0\,. \tag{3.33}$$

In summary, the plane-wave limit is given by

$$\overline{g} = 2e^{-2cx^+}dx^+dx^- + A_0\left(e^{-x^+F}\boldsymbol{x}, e^{-x^+F}\boldsymbol{x}\right)(dx^+)^2 + |d\boldsymbol{x}|^2\,, \tag{3.34}$$

where

$$\begin{aligned} c &= -S(U, U, V) \\ F_{ab} &= -S(U, Y_a, Y_b) \\ (A_0)_{ab} &= -R(U_{\mathfrak{m}}, Y_a, U_{\mathfrak{m}}, Y_b)\,, \end{aligned} \tag{3.35}$$

with the curvature given by (2.34) and the extension of S to $\mathfrak{g}$ given by

$$S(X, Y, Z) = \frac{1}{2}\langle [X, Y]_{\mathfrak{m}}, Z_{\mathfrak{m}}\rangle + \frac{1}{2}\langle [Z, X]_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle + \frac{1}{2}\langle [Z, Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle\,. \tag{3.36}$$

The result is a non-singular plane-wave if $c = 0$ and a singular plane-wave if $c \neq 0$.

Notice that the often cumbersome enterprise of taking a plane-wave limit is reduced, for the case of a homogeneous geodesic, to straightforward algebraic calculations.

3.3 The nearly-adapted method

One thing the covariant approach to plane-wave limits teaches us, is that the limit does not care about such details as the embedding of the null geodesic [9]. In particular, this means that one should be able to use a not necessarily twist-free coordinate system, which in many cases is the natural starting point, since generically a geodesic vector will not be twistfree.

[4] This is clearly consistent with its definition on $\mathfrak{m}$, as the canonical connection vanishes there. In this way it denotes the skew-symmetric endomorphism $-A_X$ of TM associated to a Killing vector, as described, for example, in [1]. Notice, though, that strictly speaking this is an abuse of notation since S is tensorial, so that $S(\mathfrak{h})$ should vanish at o but here it clearly does not.

Let γ be a null homogeneous geodesic generated by a geodetic vector $U \in \mathfrak{g}$ so that equation (2.38) holds. Let $V \in \mathfrak{m}$ be the dual null vector to $U_{\mathfrak{m}}$ and complete with $(Y_i) \in \mathfrak{m}$ to a pseudo-orthonormal frame.

Let our local coset representative σ be

$$\sigma = e^{\sum_i y^i Y_i} e^{vV} e^{uU} . \tag{3.37}$$

Then the Maurer-Cartan form θ can be expanded as

$$\sigma^*\theta = \theta^U U + \theta^V V + \theta^i Y_i + \theta^\alpha e_\alpha \tag{3.38}$$

where Greek indices are reserved for the isotropy and (e_α) is a basis for $\mathfrak{h}$. The metric can then be expanded as

$$g = 2\theta^U\theta^V + \sum_i (\theta^i)^2 . \tag{3.39}$$

Calculating the Maurer-Cartan form using σ gives

$$\sigma^*\theta = \sigma^{-1}d\sigma = e^{-uU}e^{-vV}e^{-\sum_i y^iY_i} d\left(e^{\sum_i y^iY_i}\right) e^{vV}e^{uU} + e^{-uU}V dv e^{uU} + U du . \tag{3.40}$$

A few things are clear; first du can only appear in θ^U and thus ∂_u is null. This also tells us that the isomorphism from the set of left invariant vector fields to the Lie algebra $\mathfrak{g}$ that is determined by θ maps ∂_u to U. We will denote the inverse of this isomorphism as $\mathfrak{g} \ni X \mapsto X^*$ in the following. Secondly,

$$\partial_u\theta^V = \partial_u\langle\theta_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle = U^* g(\theta^*_{\mathfrak{m}}, U^*_{\mathfrak{m}}) = g(\nabla_{U^*}\theta^*_{\mathfrak{m}}, U^*_{\mathfrak{m}}) + g(\theta^*_{\mathfrak{m}}, \nabla_{U^*}U^*_{\mathfrak{m}}) , \tag{3.41}$$

where $\theta^*_{\mathfrak{m}} = \theta^U U^* + \theta^V V^* + \theta^i Y^*_i$. Now applying the identity (2.29) we have,

$$\partial_u\theta^V = g([U^*, \theta^*_{\mathfrak{m}}], U^*_{\mathfrak{m}}) = -\langle [U, \theta_{\mathfrak{m}}]_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle = -c\langle\theta_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle = -c\theta^V , \tag{3.42}$$

where we have used that U is geodetic. This shows that the only dependence on u in θ^V is a multiplicative factor of e^{-cu}. In particular, since the dv part of θ is only dependent on u, the $dudv$ part of the metric is of the form e^{-cu}. This can be absorbed into the rest of the metric by a coordinate change:

$$u \mapsto -\frac{1}{c}\log u , \tag{3.43}$$

however, this is not necessary since u is not rescaled in the plane-wave limit. Also, it is important to note that this coordinate system is not necessarily a twist-free adapted coordinate system. We will see that this is not important and one can still take a plane-wave limit.

We can expand out the Maurer-Cartan form further and then take the plane-wave limit.

$$\theta^U = du + \langle e^{-uU} V e^{uU}, V\rangle dv + \theta^U_i dy^i , \tag{3.44}$$

where θ^U_i is a function of u, v and (y^i). Applying the plane-wave limit rescaling $(u, v, y^i) \mapsto (u, \Omega^2 v, \Omega y^i)$ to θ^U and taking the limit $\Omega \to 0$ we see that $\theta^U \to du$.

$$\theta^V = e^{-cu}\Big(dv + \Big\langle e^{-vV}e^{-\sum_i y^iY_i} d\big(e^{\sum_i y^iY_i}\big)e^{vV}, U_{\mathfrak{m}}\Big\rangle dy^i\Big)$$

$$= e^{-cu}(dv + (\langle -y^j [Y_j, Y_i]_{\mathfrak{m}} + \cdots , U_{\mathfrak{m}}\rangle dy^i) \tag{3.45}$$

where $\cdots$ are terms involving v and higher order terms in y^j. If we rescale by Ω^{-2}, apply the plane-wave limit rescaling and take the limit $\Omega \to 0$ we find that all the terms in $\cdots$ go to zero and we are left with,

$$\overline{\theta}^V = e^{-cu}(dv - y^j \langle [Y_j, Y_i]_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle dy^i)\,. \tag{3.46}$$

Similarly for θ^i we have

$$\begin{aligned}\theta^i &= \langle e^{-vV} e^{-\sum_i y^i Y_i} d\left(e^{\sum_i y^i Y_i}\right) e^{vV}, Y_j\rangle dy^j \\ &= \langle (e^{-uU} Y_j e^{uU})_{\mathfrak{m}} + \cdots , Y_i\rangle dy^j\,,\end{aligned} \tag{3.47}$$

where $\cdots$ are terms which involve v and higher order terms in y^i. Rescaling by Ω^{-1} and taking the plane-wave limit we are left with

$$\overline{\theta}^i = \langle (e^{-uU} Y_j e^{uU})_{\mathfrak{m}}, Y_i\rangle dy^j\,. \tag{3.48}$$

Therefore the plane-wave limit of the metric in this coordinate system is well defined:

$$\overline{g} = 2\overline{\theta}^V du + \sum_i (\overline{\theta}^i)^2\,. \tag{3.49}$$

Expanding this we find that the metric is nearly a plane wave in Rosen coordinates (as one would expect if this were the standard plane-wave limit) but it has an additional $dudy^i$ term with a coefficient which is linear in y^j:

$$\overline{g} = 2e^{-cu} du(dv - y^j \langle [Y_j, Y_i]_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle dy^i) + \langle (e^{-uU} Y_i e^{uU})_{\mathfrak{m}}, (e^{-uU} Y_j e^{uU})_{\mathfrak{m}}\rangle dy^i dy^j\,. \tag{3.50}$$

Note that we have had to use that U is geodetic in the calculation of the last term. We can make the change to a Brinkmann type coordinate system irrespective of this extra term. We find the metric in Brinkmann coordinates is

$$\begin{aligned}\overline{g} = {}& 2e^{-2cx^+} dx^- dx^+ - (\langle [U, Y_a]_{\mathfrak{m}}, Y_b\rangle - \langle [U, Y_b]_{\mathfrak{m}}, Y_a\rangle - \langle [Y_a, Y_b]_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle) x^b dx^a dx^+ \\ & + (\langle [U, Y_a]_{\mathfrak{m}}, [U, Y_b]_{\mathfrak{m}}\rangle - \langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U\rangle) x^a x^b (dx^+)^2 + \sum_i (dx^i)^2\,.\end{aligned} \tag{3.51}$$

Notice that

$$\langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U\rangle \tag{3.52}$$

is symmetric in a and b because of the Jacobi identity and the geodetic vector property (2.38). In light of the above, we also define

$$F_{ab} = \frac{1}{2}\langle [Y_a, Y_b]_{\mathfrak{m}}, U_{\mathfrak{m}}\rangle - \frac{1}{2}\langle [U, Y_a]_{\mathfrak{m}}, Y_b\rangle + \frac{1}{2}\langle [U, Y_b]_{\mathfrak{m}}, Y_a\rangle\,. \tag{3.53}$$

To show this is a plane wave and bring it to the proper Brinkmann form we make the change of coordinates $\boldsymbol{x} \mapsto e^{-x^+ F}\boldsymbol{x}$, which leaves the metric in the form

$$2e^{-2cx^+}dx^-dx^+ + A_0\left(e^{-x^+F}\boldsymbol{x}, e^{-x^+F}\boldsymbol{x}\right)(dx^+)^2 + |d\boldsymbol{x}|^2\,, \tag{3.54}$$

where

$$(A_0)_{ab} = \langle [U, Y_a]_{\mathfrak{m}}, [U, Y_b]_{\mathfrak{m}}\rangle - \langle [Y_a, [U, Y_b]]_{\mathfrak{m}}, U\rangle + F_{ab}^2\,. \tag{3.55}$$

An easy check shows that these formulae do indeed coincide with those derived by the covariant method.

However, since we have not worked with an adapted coordinate system, at no stage in the above discuss have we proved that the formula we have obtained is actually applicable to the usual plane-wave limit of the geodesic γ. We will now remedy this situation.

Consider a metric of the form

$$2dudv + \alpha dv^2 + \beta_i dy^i dv + K_{ij}y^i dy^j du + C_{ij}dy^i dy^j\,, \tag{3.56}$$

such that ∂_u is a null geodesic and K_{ij} is skew-symmetric. Up to a coordinate transformation in u this is the form of the metric in equation (3.39). An easy calculation shows that the R_{uiuj} component of the curvature of this metric:

$$R(\partial_u, \partial_i)\partial_u = -\nabla_{\partial_u}\nabla_{\partial_i}\partial_u + \nabla_{\partial_i}\nabla_{\partial_u}\partial_u + \nabla_{[\partial_u,\partial_i]}\partial_u\,, \tag{3.57}$$

is independent of K_{ij}. If we apply the plane-wave limit rescaling, multiply by Ω^{-2} and take the limit as $\Omega \to 0$ we get

$$2dudv + K_{ij}y^i dy^j du + C_{ij}(u)dy^i dy^j\,. \tag{3.58}$$

This metric is a plane wave, as we can change to Brinkmann coordinates and then absorb the linear term into the rest of the metric (as we did above). Since a plane wave is completely determined by the R_{uiuj} part of its curvature, the metric (3.54) must be isometric to the usual plane-wave limit of the geodesic ∂_u.

Let us consider in passing the plane-wave limits of the WZW model. The algebraic data of a WZW model consists of a Lie algebra $\mathfrak{g}$ together with an nondegenerate invariant inner product. The geometry is therefore trivially that of a naturally reductive space which, as we have seen above, means that every null vector in $\mathfrak{g}$ gives rise to a homogeneous geodesic, hence every plane-wave limit preserves homogeneity. Moreover plane-wave limits are equivalent to contracting $\mathfrak{g}$. The above method for calculating the plane-wave limit is just a manifestation of this fact: instead of expanding the Maurer-Cartan forms in Ω, one can redefine the generators $\widetilde{U} = U$, $\widetilde{V} = \Omega^2 V$ and $\widetilde{E}_a = \Omega E_a$ with a new inner product $\langle \widetilde{U}, \widetilde{V}\rangle' = 1$ and $\langle \widetilde{E}_a, \widetilde{E}_b\rangle' = \eta_{ab}$. This then means that we can write down a family of WZW models in terms of $\langle -, -\rangle'$ and the Lie bracket — equivalently, the homogeneous structure — that interpolates between the original model at $\Omega = 1$ and its plane-wave limit at $\Omega = 0$. It is then clear that the Poisson bracket for this family has a regular limit, and that group contraction is extended to a contraction of the associated affine algebra and also of the Yangian [25]. This means that the diagram of contraction and quantisation in [26] should indeed commute.

3.4 Homogeneous structures under the plane-wave limit

There are circumstances where one can say more than that the plane-wave limit is homogeneous, and in addition say something about the type of homogeneous structure inherited by the plane-wave limit along a homogeneous geodesic. It is clear from (3.35) that the homogeneous structure of the plane-wave limit along a homogeneous geodesic is inherited from the original metric g in some sense, since the whole plane-wave limit metric is defined in terms of the algebraic data. In fact, (3.35) for F can be interpreted as the Ambrose-Singer formula $\overline{\nabla}\,\overline{R} = \overline{S} \cdot \overline{R}$ on the plane-wave limit. However, this homogeneous structure is not inherited continuously in the limit, so it is difficult to draw conclusions about the type of homogeneous structure inherited under the plane-wave limit. To study this situation we may consider a stronger form of heritability of the homogeneous structure, namely when this is inherited continuously in the limit.

On a reductive homogeneous space, the Ambrose-Singer theorem provides us with a connection, namely $\widetilde{\nabla}$, relative to which the metric g, the Riemann tensor R and the homogeneous structure S are parallel. One way to guarantee the heritability of homogeneity would be to have a well-defined limit of $\widetilde{\nabla}$ or rather, since ∇ has a well-defined limit, to have a well-defined limit of the homogeneous structure.

When $\widetilde{\nabla}$ has a well-defined limit, then also its curvature has a well-defined limit and, seeing the discussion around equation (2.27), one must conclude that the plane-wave limit is equivalent to a Inönü-Wigner contraction, where the extra isometries that can arise through the plane-wave limit will be elements of the isotropy subalgebra. A result mentioned in [7] gives a sufficient and necessary criterion for this to happen: there exists a well-defined plane-wave limit of S if and only if the geodesic along which the limit is performed is canonically homogeneous. Since on a symmetric space all geodesics are canonically homogeneous, this provides an *a posteriori* explanation for the results in [27]. Furthermore, on the Kaigorodov space there are two canonically homogeneous geodesics, and as such the identification in [6, section 3.3], albeit through a different contraction, can be reproduced. One can easily check that this leads to the statement that the resulting singular plane wave is also a group manifold with a left-invariant metric.

Let us then have a better look at the plane-wave limit of S. Since the torsion in the plane-wave limit must scale as the Levi-Cività connection, it is evident that the plane-wave limit of S is given by

$$\overline{S} = \lim_{\Omega\to 0} \Omega^{-2} \frac{1}{2} S(\tilde{y})_{ABC} d\tilde{y}^A \otimes d\tilde{y}^B \wedge d\tilde{y}^C \,, \tag{3.59}$$

where $\tilde{y} = (u, \Omega^2 v, \Omega y^i)$. It then follows that the condition for the existence of a non-singular limit is the regularity of $\lim_{\Omega\to 0} \Omega^{-1} S_{uui}(\tilde{y})$, which together with S_{uuv}, S_{uij} and S_{iuj}, are the components surviving in the plane-wave limit.

We are now in a position to see why the regularity of the plane-wave limit implies that the geodesic must be canonically homogeneous. Indeed, if we let U be the geodetic vector, so that $\nabla_U U = 0$, then

$$S_{uui} = S(U, U, \partial_i) = -g(\partial_i, \widetilde{\nabla}_U U) \,. \tag{3.60}$$

Decomposing $\widetilde{\nabla}_U U = AU + B^i \partial_i$, where a ∂_v contribution is impossible due to the fact that $S(U,U,U) = 0$, one sees that $S_{uui} = -C_{ij} B^j$. This implies that $B^j = 0$ if and only if a regular plane-wave limit of S exists. At the same time this implies $\widetilde{\nabla}_U U \propto U$, or rather the geodesic is also a geodesic of the canonical connection, whence the geodesic is canonically homogeneous.

A first thing to observe is that the component S_{uui} cannot be part of $\mathcal{T}_1$ nor of $\mathcal{T}_3$, from which we can conclude that the plane-wave limit of a $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure is regular. This reinforces the discussion about the existence of canonically homogeneous geodesics on spaces admitting a $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure in section 3.1.

We can be a bit more precise as to what homogeneous structure the plane-wave limit of a $\mathcal{T}_1 \oplus \mathcal{T}_3$ space will inherit, by applying the formula (3.35). Consider a null geodesic γ generated by the Killing vector $U \in \mathfrak{m}$ and dual null vector $V \in \mathfrak{m}$. Then, introducing $\alpha(Z) = g(\xi, Z)$ as in equation (3.35),

$$c = -S(U,U,V) = \alpha(U)\,. \tag{3.61}$$

There are two scenarios to consider, i) $\alpha(U) = 0$ and ii) $\alpha(U) \neq 0$. The first case corresponds to the original space g being naturally reductive and the second case to g being a singular homogeneous plane-wave. Comparing this with the classification of homogeneous plane-waves reviewed in section 2.5, we must conclude that in case i) the resulting spacetime admits a pure $\mathcal{T}_3$ structure and must be a regular homogeneous plane wave, whereas in case ii) the resulting spacetime is a singular homogeneous plane wave. Note that the plane-wave limit of a plane-wave is not necessarily trivial, only if the limit is along the defining null geodesic of the wave.

In general not much more can be said about which classes of reductive homogeneous spaces guarantee the existence of canonically homogeneous null geodesics. This is exemplified by the next three examples, all of which are discussed in more detail in section 4. The first example is of course the Kaigorodov space, for which there exists a unique homogeneous structure of generic type, for example, equation (4.6). Evaluating this homogeneous structure in the plane-wave limit one can see that it is well defined if and only if the limit is taken along a null geodesic with initial direction given by (4.14) with $\alpha^2 = 1$. These are just the cases that characterise the null homogeneous geodesics and lead to a singular homogeneous plane wave or flat space.

The above example shows that if a spacetime has a unique homogeneous structure containing a $\mathcal{T}_2$ contribution, then generically there are geodesics such that homogeneity is lost in the plane-wave limit. The next example shows that this also holds for cases where we have a family of homogeneous structures. Consider the Komrakov 1.4.6 metric [10]

$$e^{-2y}\left(2du\,[dv + y du] + dx^2\right) + dy^2\,. \tag{3.62}$$

As shown in section 4.5.2, this homogeneous metric admits a two-parameter family of generic homogeneous structures, which minimally is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$. The only plane-wave limit for which the resulting spacetime is homogeneous is in the v direction, which incidentally also corresponds to the only (canonically) homogeneous null geodesic, leading to flat space.

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Seeing the above examples, one might be tempted to conclude that if the homogeneous structure contains a $\mathcal{T}_2$ contribution, then there are geodesics such that the plane-wave limit is non-homogeneous. In order to show that this is certainly not the case, consider the metric (see section 4.4)

$$ds^2 = -\left(dt - x^1 dx^4 - x^2 dx^3\right)^2 + \left(dy + x^2 dx^4 - x^1 dx^3\right)^2 + dx_i dx^i\,, \tag{3.63}$$

which is a lorentzian version of Kaplan's first example of a g.o. space, that is in no way naturally reductive (see, for example, [18, 28]). A small calculation of the homogeneous structure shows that it admits a 3-parameter family of $\mathcal{T}_2 \oplus \mathcal{T}_3$ structures, which does not contain a pure $\mathcal{T}_3$ point. Since this space is g.o., every (null) geodesic is homogeneous and we are guaranteed that the plane-wave limit is homogeneous. There are however only two canonically homogeneous null geodesics, namely those along $X_5 \pm X_6$, and for those the resulting spacetime is flat space and the limit of S is regular. For the remaining null geodesics, the result of the plane-wave limit is a homogeneous plane wave, but the plane-wave limit of S is not well-behaved; this is due to the fact that although the null geodesics are homogeneous, there is no reductive split such that a given null geodesic becomes canonical.

The conclusion of the examples then must be that on a reductive homogeneous space whose homogeneous structure always contains a $\mathcal{T}_2$ contribution, there exist plane-wave limits along which the homogeneous structure is singular. This, however, need not imply the absence of homogeneous geodesics and thus loss of homogeneity in the plane-wave limit, as evidenced by the example of Kaplan's g.o. space.

Given a reductive homogeneous space (M, g) which admits a homogeneous structure of a type other than $\mathcal{T}_1 \oplus \mathcal{T}_3$ and given a canonically homogeneous geodesic γ of (M, g), what can one say about the homogeneous plane-wave limit along γ? Let us consider the other two types of homogeneous structures, $\mathcal{T}_1 \oplus \mathcal{T}_2$ and $\mathcal{T}_2 \oplus \mathcal{T}_3$, separately. First, suppose that g admits a homogeneous structure S of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, that is S is in the kernel of the map C defined in (2.17). Then the plane-wave rescaling S_Ω will be in the kernel of C_Ω, and hence a homogeneous structure of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ for g_Ω (as defined in section 3.1). If we can define a family of pseudo orthonormal frames $(e_a(\Omega))$ for g_Ω, such that in the limit $\Omega \to 0$ the frame $(e_a(0))$ is a well-defined pseudo orthonormal frame for $\overline{g}$, then continuity will ensure that $\overline{S}$ is in the kernel of $\overline{C}$.

We can exhibit such a basis by applying the Gram-Schmidt process to the frame $(\partial_{y^1}, \ldots, \partial_{y^{n-2}}, \partial_u - \partial_v, \partial_u + \partial_v)$ for (3.1), starting from the left. This gives an orthonormal frame $(e_1(\Omega), \ldots, e_{n-2}(\Omega))$ for the transverse y^i coordinates, together with

$$\begin{aligned}
e_u(\Omega) &= \frac{1}{(2 - \Omega^2\alpha - \sum_{i=1}^{n-1} \Omega^2 \eta_i^2)} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v} + \Omega \sum_{i=1}^{n-1} \eta_i e^i \right), \\
e_v(\Omega) &= \frac{1}{(2 + \Phi(\Omega))} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \Omega^2 \alpha e_u(\Omega) - \Omega \sum_{i=1}^{n-1} (\eta_i e^i + \Omega^2 \eta_i^2 e^u) \right),
\end{aligned} \tag{3.64}$$

where $\eta_i := g(\frac{\partial}{\partial v}, e^i)$ and $\Phi(\Omega)$ is some function of (Ω, u, v, y^k) such that $|e^v|^2 = 1$ and which tends to zero as $\Omega \to 0$. Taking the limit $\Omega \to 0$, we obtain an orthonormal basis with respect to $\overline{g}$, namely $(e_1, \ldots, e_{n-2}, (\partial_u - \partial_v)/2, (\partial_u + \partial_v)/2)$.

Consequently, we find that $\overline{S}$ is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. Since homogeneous plane-waves are essentially of type $\mathcal{T}_1 \oplus \mathcal{T}_3$, this gives a further restriction on $\overline{S}$:

$$0 = C(\overline{S})(\partial_v) = \overline{S}(e_u, e_u, \partial_v) + \overline{S}(e_v, e_v, \partial_v) + \overline{S}(e_i, e_i, \partial_v) = \frac{1}{2}\overline{S}_{uuv}\,, \tag{3.65}$$

where we have used $\overline{S}^u_{vv} = \overline{S}^u_{e_ie_i} = 0$. We have already seen (2.44) that $\overline{S}_{uuv}$ is non-zero for the singular plane-waves. Therefore, the plane-wave limit must be a non-singular plane-wave.

If g admits a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_2$, so that $S^{\boxminus}$ vanishes, then one can merely say that the plane-wave limit also admits a homogeneous structure of this type (by continuity). This homogeneous structure $\overline{S}$ may not be of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ since the plane-wave may admit many different homogeneous structures. In addition, the formulae (3.35) shed little light on the subject.

4. Examples

In this section we discuss several homogeneous spacetimes in detail and discuss their homogeneous structures. We also compute their plane-wave limits using our Lie algebraic formulation. First we summarise the methodology by which we explore the possible plane-wave limits.

4.1 Methodology

Given a homogeneous space in terms of the following data: a Lie algebra of isometries $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with an $\mathfrak{h}$-invariant lorentzian inner product $\langle -, - \rangle$ on $\mathfrak{m}$, we follow the following procedure:

1. We first determine the orbit decomposition of the projectivised light-cone of $\mathfrak{m}$ under the exponentiated action of $\mathfrak{h}$. This will determine the possible null directions up to isometry. In practise we label these orbits by giving a null direction in each orbit.

2. For each such null direction $u \in \mathfrak{m}$ we determine whether the null geodesic pointing along u is homogeneous. In other words, we determine whether there is some $X \in \mathfrak{h}$ for which $U := u + X$ is geodetic; that is, whether U obeys (2.38) for some value of c. If it does, then the plane-wave limit along U will be homogeneous: regular if $c = 0$ and singular otherwise.

3. Finally we determine the explicit form of the plane-wave metric. To do this we choose a frame u, V, Y_a for $\mathfrak{m}$ such that $\langle u, V \rangle = 1$ and $\langle Y_a, Y_b \rangle = \delta_{ab}$ and then compute the matrices F and A using formulae (3.53) and (3.55), respectively.

Clearly many of these steps can be performed (or at least checked) using one's favourite computer algebra software.

4.2 Kaigorodov spaces

The Kaigorodov space K is an $(n+3)$-dimensional lorentzian manifold with metric [29]

$$-(\theta^0)^2 + \sum_{i=1}^{n+2} (\theta^i)^2 \tag{4.1}$$

where

$$\theta^0 = e^{(4+n)\ell\rho} dt\,, \qquad \theta^i = e^{2\ell\rho} dy^i\,, \qquad \theta^{n+1} = e^{-n\ell\rho} dx + e^{(4+n)\ell\rho} dt\,, \qquad \theta^{n+2} = d\rho\,, \tag{4.2}$$

where, here and in the sequel, the indices $i, j, \ldots$ run from 1 to n. This spacetime can be seen to have a pp-wave singularity and is not geodesically complete [30]. Up to homothety, we can (and will) set $\ell = 1$ from now on.

We observe that the θ^a form a differential ideal with structure *constants*:

$$\begin{aligned} d\theta^0 &= (4+n)\theta^{n+2} \wedge \theta^0 \\ d\theta^i &= 2\theta^{n+2} \wedge \theta^i \\ d\theta^{n+1} &= -n\theta^{n+2} \wedge \theta^{n+1} + 2(2+n)\theta^{n+2} \wedge \theta^0 \\ d\theta^{n+2} &= 0\,. \end{aligned} \tag{4.3}$$

This means that the dual vector fields X_a, defined by $\theta^a(X_b) = \delta^a_b$, form a Lie algebra, denoted $\mathfrak{k}$. We can read off their Lie brackets from the above differentials

$$\begin{aligned} [X_{n+2}, X_0] &= -(4+n)X_0 - 2(2+n)X_{n+1} \\ [X_{n+2}, X_{n+1}] &= nX_{n+1} \\ [X_{n+2}, X_i] &= -2X_i\,. \end{aligned} \tag{4.4}$$

Notice that from the expression of the metric, these vector fields form a pseudo-orthonormal frame. It is convenient to diagonalise the adjoint action of X_{n+2} by redefining $X_0 \mapsto X_0 + X_{n+1}$. We now have the simpler brackets

$$\begin{aligned} [X_{n+2}, X_0] &= -(4+n)X_0 \\ [X_{n+2}, X_{n+1}] &= nX_{n+1} \\ [X_{n+2}, X_i] &= -2X_i\,, \end{aligned} \tag{4.5}$$

at the price that the metric in this new basis is no longer diagonal, but instead X_0 is now null and $\langle X_0, X_{n+1}\rangle = 1$.

In summary, we have exhibited the Kaigorodov space K as a Lie group with a left-invariant lorentzian metric. In particular it is trivially reductive. The corresponding homogeneous structure $S_{abc} = S(X_a, X_b, X_c)$, from equation (2.32), is given by

$$\begin{aligned} S_{n+2,0,n+1} &= -(2+n) & \qquad S_{n+1,0,n+2} = S_{0,n+1,n+2} &= -2 \\ S_{n+1,n+1,n+2} &= n & S_{i,j,n+2} &= -\delta_{ij}\,. \end{aligned} \tag{4.6}$$

It is not hard to see that it has generic type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

The full isometry Lie algebra of K is larger than the Lie algebra $\mathfrak{k}$ generated by the X_a. Indeed, it has in addition an $\mathfrak{iso}(n)$ Lie algebra, with generators L_{ij} and L_i, with the $\mathfrak{so}(n)$ generators L_{ij} acting on the X_i as vectors and together with the following brackets

$$\begin{aligned} [L_i, X_j] &= -\delta_{ij} X_0 \\ [L_i, X_{n+1}] &= X_i \\ [X_{n+2}, L_i] &= -(n+2) L_i \,. \end{aligned} \tag{4.7}$$

It is clear from this last bracket that this is not a reductive split; although we still have an action of $\mathfrak{iso}(n)$ on the tangent space $T_o K$ at the identity by projecting the Lie bracket to $\mathfrak{k}$.

We now determine the action of the isotropy group $\mathrm{ISO}(n)$ on the celestial sphere in $T_o K$. Relative to the ordered basis $(X_1, \ldots, X_{n+2}, X_0)$, the typical element $(A, \boldsymbol{b})$ of $\mathrm{ISO}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ has matrix

$$\begin{pmatrix} A & A\boldsymbol{b} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 1 & 0 & 0 \\ \boldsymbol{0} & 0 & 1 & 0 \\ -\boldsymbol{b}^t & -\frac{1}{2}|\boldsymbol{b}|^2 & 0 & 1 \end{pmatrix} \tag{4.8}$$

which has been obtained as the product

$$\begin{pmatrix} A & \\ & I_3 \end{pmatrix} \exp(b^i L_i) \,. \tag{4.9}$$

Acting on a tangent vector $v = (\boldsymbol{v}, v^{n+1}, v^{n+2}, v^0) \in T_o K$, we find

$$(A, \boldsymbol{b}) \cdot \begin{pmatrix} \boldsymbol{v} \\ v^{n+1} \\ v^{n+2} \\ v^0 \end{pmatrix} = \begin{pmatrix} A\boldsymbol{v} + v^{n+1} A\boldsymbol{b} \\ v^{n+1} \\ v^{n+2} \\ v^0 - \boldsymbol{b}^t \boldsymbol{v} - \frac{1}{2}|\boldsymbol{b}|^2 v^{n+1} \end{pmatrix} . \tag{4.10}$$

Let v have zero norm, so that

$$(v^{n+1})^2 + (v^{n+2})^2 + |\boldsymbol{v}|^2 = -2 v^0 v^{n+1} \,. \tag{4.11}$$

Since $v \neq 0$, it follows that $v^0 \neq 0$. We must therefore distinguish two cases, according to whether v^{n+1} does or does not vanish.

- If $v^{n+1} = 0$, then also $v^{n+2} = 0$ and $\boldsymbol{v} = \boldsymbol{0}$. We can then choose $v^0 = 1$, whence $v = X_0$.

- If $v^{n+1} \neq 0$, then we can choose $\boldsymbol{b} = -\boldsymbol{v}/v^{n+1}$ to bring v to the form

$$\begin{pmatrix} \boldsymbol{0} \\ v^{n+1} \\ v^{n+2} \\ -\frac{1}{2v^{n+1}}\left((v^{n+1})^2 + (v^{n+2})^2\right) \end{pmatrix} , \tag{4.12}$$

where we have used that v is null. We can choose $v^{n+1} = 1$, $v^{n+2} = \alpha$ so that finally

$$v = X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0 \,. \tag{4.13}$$

In summary, we have two possible null directions up to the action of the isotropy subgroup, one of them parametrised by a real number α:

$$X_0 \qquad \text{and} \qquad X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0 \,. \tag{4.14}$$

One checks that X_0 is a geodetic vector with $c = 0$, and that the null geodesic along $X_{n+1} + \alpha X_{n+2} - \frac{1}{2}(1+\alpha^2)X_0$ is homogeneous only when $\alpha^2 = 1$, in which case $X_{n+1} + \alpha X_{n+2} - X_0$ is geodetic with $c = -\alpha(4+n)$. In the first case, therefore, the corresponding plane-wave limits will be a regular homogeneous plane wave, whereas in the second case the limit will be a singular homogeneous plane wave.

It is not difficult to see that in both cases the skew-symmetric matrix F given by equation (3.53) vanishes. This means the limit is a symmetric plane wave, whose metric is determined by the symmetric matrix A in equation (3.55). It is easy to show that in the first case, where the geodetic vector is X_0, the symmetric matrix $A = 0$, whence the plane-wave limit is flat. In the case where the geodetic vector is $X_{n+1} + \alpha X_{n+2} - X_0$, a calculation shows that the nonzero components of A are

$$A_{ij} = 4\delta_{ij} \qquad \text{and} \qquad A_{n+1,n+1} = n^2 \,. \tag{4.15}$$

4.3 Higher-dimensional Gödel universes

The five-dimensional Gödel universe is a reductive spacetime and also a maximally supersymmetric solution of minimal five-dimensional supergravity, whose M-theory lift preserves 20 supersymmetries [31]. The plane-wave limit of the five-dimensional Gödel universe is the five-dimensional maximally supersymmetric plane wave [32]. The plane-wave limits of the M-theory Gödel universe were investigated in [33] and shown to give rise to a family of time-dependent plane waves interpolating between two Cahen-Wallach spaces, one of which corresponds to the M-theory lift of the five-dimensional maximally supersymmetric plane wave. In this subsection, we will rederive these results using our Lie algebraic formalism.

4.3.1 The five-dimensional Gödel universe

We start with the five-dimensional Gödel universe, which is defined on a circle bundle over flat euclidean space:

$$g = -(dt + A)^2 + \sum_{i=1}^{4}(dx^i)^2 \,, \tag{4.16}$$

where the connection one-form A is given by

$$A = \frac{1}{2}(x^1dx^2 - x^2dx^1) - \frac{1}{2}(x^3dx^4 - x^4dx^3) \,. \tag{4.17}$$

The two-form F is given simply by

$$F = dA = dx^1 \wedge dx^2 - dx^3 \wedge dx^4 \,, \tag{4.18}$$

which is clearly an anti-selfdual two-form in $\mathbb{E}^4$ with respect to the natural orientation. Any infinitesimal symmetry of F can be promoted to an isometry by adding a compensating gauge transformation. The two-form F is manifestly invariant under a subgroup $\mathrm{U}(2) \ltimes \mathbb{R}^4$ of the euclidean group of isometries of $\mathbb{E}^4$ and in addition by the $\mathrm{U}(1)$ group of translations along the fibre generated infinitesimally by ∂_t. Clearly $\mathrm{U}(2)$ and the fibre $\mathrm{U}(1)$ are still invariances of the metric, but the translations are not because they do not leave A invariant. Nevertheless they can be corrected to make $dt + A$ and hence the metric invariant. Doing so one finds the following Killing vectors leaving g and F invariant:

$$\begin{gathered}
\partial_t \qquad \partial_1 - \frac{1}{2}x^2\partial_t \qquad \partial_2 + \frac{1}{2}x^1\partial_t \qquad \partial_3 + \frac{1}{2}x^4\partial_t \qquad \partial_4 - \frac{1}{2}x^3\partial_t \\
x^1\partial_2 - x^2\partial_1 \qquad x^3\partial_4 - x^4\partial_3 \\
x^1\partial_3 - x^3\partial_1 + x^2\partial_4 - x^4\partial_2 \qquad x^1\partial_4 - x^4\partial_1 - x^2\partial_3 + x^3\partial_2 \,.
\end{gathered} \tag{4.19}$$

Notice that at any point (t, x^i) of M, the five Killing vectors in the first line span the tangent space, so that M is indeed a homogeneous space.

The isometry algebra $\mathfrak{g}$ is isomorphic to the semidirect product

$$\mathfrak{g} \cong (\mathfrak{su}(2) \times \mathfrak{u}(1)) \ltimes \mathfrak{h}(4)\,, \tag{4.20}$$

where $\mathfrak{h}(4)$ is the five-dimensional Heisenberg algebra

$$[P_i, P_j] = \Omega_{ij} P_0\,, \tag{4.21}$$

generated by $P_0 = \partial_t$ and $P_i = \partial_i - \frac{1}{2}\sum_j \Omega_{ij} x^j \partial_t$, where Ω_{ij} is the symplectic form with nonzero entries $\Omega_{12} = 1 = -\Omega_{21}$ and $\Omega_{34} = -1 = -\Omega_{43}$. In the above expression for $\mathfrak{g}$, $\mathfrak{su}(2) \times \mathfrak{u}(1) \subset \mathfrak{so}(4)$ acts on $\mathfrak{h}(4)$ by restricting the natural action of $\mathfrak{so}(4)$ under which P_0 is a scalar and P_i is a vector. The corresponding isometry group G is given by

$$G \cong \mathrm{U}(2) \ltimes \mathrm{H}(4)\,, \tag{4.22}$$

with $\mathrm{U}(2) \subset \mathrm{SO}(4)$ acting on $\mathrm{H}(4)$ in the natural way.

Let $o \in M$ be the point with coordinates $(t = x^i = 0)$. The vectors $P_0, P_1, \ldots, P_4$ form a pseudo-orthonormal frame for T_oM, with P_0 timelike. The little group of o is precisely the natural $\mathrm{U}(2)$ subgroup of G. Therefore $M = G/\mathrm{U}(2)$ is the G-orbit of o. From the classification of lorentzian symmetric spaces in [34] or from a direct calculation, it follows that M is not symmetric.

We can also see this by exhibiting the homogeneous structures of the Gödel universe. Considering the reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{g}$ the full isometry algebra, we find using equation (2.32) that the components $S_{abc} = S(P_a, P_b, P_c)$ of the homogeneous structure at o are given by

$$S_{0ij} = S_{i0j} = -S_{ij0} = \frac{1}{2}\Omega_{ij}\,, \tag{4.23}$$

which can be seen to be of type $\mathcal{T}_2 \oplus \mathcal{T}_3$.

We can deform this homogeneous structure by considering a reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ where $\mathfrak{m}'$ is the graph of an $\mathfrak{h}$-equivariant linear map $\mathfrak{m} \to \mathfrak{h}$. Decomposing $\mathfrak{m}$ and $\mathfrak{h}$ into irreducibles we find that there is a one-parameter map of such linear maps $\varphi_\alpha(v^i P_i) = \alpha v^0 Y_0$, where $Y_0 \in \mathfrak{h}$ is the Killing vector $Y_0 = x^1\partial_2 - x^2\partial_1 + x^3\partial_4 - x^4\partial_3$. Its graph $\mathfrak{m}'$ is spanned by

$$P_1\,, \quad P_2\,, \quad P_3\,, \quad P_4\,, \quad \text{and} \quad P_0 + \alpha Y_0\,. \tag{4.24}$$

This modifies the $[-,-]_{\mathfrak{m}'}$ brackets:

$$[P_i, P_j]_{\mathfrak{m}'} = \Omega_{ij}(P_0 + \alpha Y_0) \qquad \text{and} \qquad [P_0 + \alpha Y_0, P_i]_{\mathfrak{m}'} = \alpha \Omega_{ij} P_j\,. \tag{4.25}$$

We can now compute the corresponding homogeneous structure using formula (2.32) and we obtain a one-parameter family of $\mathcal{T}_2 \oplus \mathcal{T}_3$ structures:

$$S_{0ij} = \left(\frac{1}{2} + \alpha\right)\Omega_{ij} \qquad \text{and} \qquad S_{i0j} = -S_{ij0} = \frac{1}{2}\Omega_{ij}\,. \tag{4.26}$$

Naturally, when $\alpha = 0$ we recover the earlier homogeneous structure. Clearly for generic α we have a homogeneous structure of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, but for $\alpha = -1$ it is of type $\mathcal{T}_3$ and for $\alpha = 1/2$ it is of type $\mathcal{T}_2$.

One can obtain more homogeneous structures by considering smaller subalgebras, but we will not do so here.

In order to determine the possible plane-wave limits of the Gödel universe we will exploit the covariance property of the plane-wave limit [5]. This says that if two null geodesics in M are related by an isometry of M, then the corresponding plane-wave limits are themselves isometric. A null geodesic γ in M is locally determined by the following data: an initial point $\gamma(0) \in M$ and an initial direction $\gamma'(0)$, which is a point on the future-pointing, say, celestial sphere at $\gamma(0)$. Since M is homogeneous, we can choose $\gamma(0)$ to be any convenient point. We will choose the point o above with coordinates $(t = 0, x^i = 0)$ and retain the freedom of using the isotropy subgroup of o. The (future) celestial sphere at o consists of those vectors $v = v^\mu P_\mu$ such that $\langle v, v\rangle = 0$ and $v^0 = 1$, which is the unit three-sphere in $\mathbb{E}^4 = \langle P_0\rangle^\perp$. The isotropy group U(2) acts on $\mathbb{E}^4$ by restricting the natural representation of SO(4), whence it acts transitively on the spheres. Therefore we see that the isometry group of (M, g, F) acts transitively on the space of null geodesics and hence all plane-wave limits are isometric.

Let us choose our geodesic to point in the direction of $P_0 + P_1$. This vector is not geodetic, since it does not satisfy equation (2.38) for any value of c. We modify it by adding a vector $X \in \mathfrak{h}$ in such a way that (2.38) is satisfied. A quick calculation shows that $X = -Y_0$ does the trick. The resulting Killing vector $P_0 + P_1 - Y_0$ is geodetic with $c = 0$. This means that the plane-wave limit will be a regular homogeneous wave. Moreover, we see that this is a canonically homogeneous geodesic, since as shown above there is a reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ with $\mathfrak{m}'$ spanned by $P_0 - Y_0$, P_1, P_2, P_3 and P_4.

In fact, as we now show, the limit is a symmetric plane wave, with geometry a five-dimensional Cahen-Wallach space. This vacuum of minimal five-dimensional supergravity was discovered in [32]. To determine the limit we employ the formulae (3.53) and (3.55). We find that

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \text{and} \qquad A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}, \tag{4.27}$$

in agreement with the results of [33].

4.3.2 The Gödel universe in M-theory

The five-dimensional Gödel universe can be lifted to a supersymmetric M-theory background $(\widetilde{M}, g, G)$ preserving 20 supersymmetries [31] simply by taking its cartesian product with a flat six-dimensional space. It is convenient to think of this six-dimensional space as $\mathbb{C}^3$ with its standard Kähler structure ω, whence $\widetilde{M} = M \times \mathbb{C}^3$ metrically. The M-theory four-form is then $G = F \wedge \omega$, whence the symmetry group of this M-theory background is

$$(\mathrm{U}(2) \ltimes \mathrm{H}(4)) \times \left(\mathrm{U}(3) \ltimes \mathbb{R}^6\right), \tag{4.28}$$

which still acts transitively, making $(\widetilde{M}, g, G)$ into a homogeneous background. Let z^α denote local coordinates on $\mathbb{C}^3$ and let o be the point on the eleven-dimensional product manifold with coordinates $t = x^i = z^\alpha = 0$. The isotropy subgroup at this point is $\mathrm{U}(2) \times \mathrm{U}(3)$, which is reductive.

The isotropy subgroup acts with cohomogeneity one in the (future) celestial sphere in $T_o\widetilde{M}$. Indeed, any tangent vector decomposes into $\boldsymbol{v} = \boldsymbol{v}_G + \boldsymbol{v}'$, with $\boldsymbol{v}_G$ the component tangent to the five-dimensional Gödel universe and $\boldsymbol{v}'$ the component tangent to $\mathbb{C}^3$. The isotropy subgroup preserves the norms $|\boldsymbol{v}_G|^2$ and $|\boldsymbol{v}'|^2$ separately. Let $\boldsymbol{v}$ be a future-pointing null vector. By further rescaling, we can ensure that its P_0 component is 1, whence $\boldsymbol{v}_G = P_0 + \boldsymbol{v}_\perp$ where $|\boldsymbol{v}_\perp|^2 + |\boldsymbol{v}'|^2 = 1$. Fix an angle $\vartheta \in [0, \frac{\pi}{2}]$ and let $|\boldsymbol{v}_\perp| = \cos\vartheta$ and $|\boldsymbol{v}'| = \sin\vartheta$. The isotropy subgroup cannot change ϑ, but it acts transitively on these spheres, whence we can make $\boldsymbol{v}_\perp$ and $\boldsymbol{v}'$ point in any desired direction. Letting T_i denote the translation generators for the $\mathbb{R}^6$ subgroup of the isometries of $\mathbb{C}^3$, we can write the null vector as

$$P_0 + \cos\vartheta P_1 + \sin\vartheta T_1 \,. \tag{4.29}$$

This vector is not geodetic, however, unless we add $-Y_0$, as in the five-dimensional Gödel universe. Doing so we see that

$$P_0 + \cos\vartheta P_1 + \sin\vartheta T_1 - Y_0 \tag{4.30}$$

does obey equation (2.38) with $c = 0$. This means that the plane-wave limits will again be regular.

Indeed, using equation (3.53), we find that the only nonzero components of F are

$$F_{14} = -\frac{1}{2}\sin\vartheta \qquad \text{and} \qquad F_{23} = -\frac{1}{2}\,. \tag{4.31}$$

Similarly, using equation (3.55) the matrix A has nonzero components

$$A_{11} = -1 + \frac{3}{4}\sin^2\vartheta\,, \quad A_{22} = A_{33} = -\frac{1}{4}\,, \qquad \text{and} \qquad A_{44} = -\frac{1}{4}\sin^2\vartheta\,. \tag{4.32}$$

Notice that since $[A, F] \neq 0$ this is not a symmetric plane wave.

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4.4 A lorentzian g.o. space

In this section we will discuss the geometry of a six-dimensional lorentzian g.o. space which is not naturally reductive.

4.4.1 The geometry

The lorentzian version of Kaplan's g.o. space (see, for example, [18]) is a six-dimensional 2-step nilpotent Lie group M with a left invariant metric. The Lie algebra $\mathfrak{m}$ is spanned by X_i for $i = 1, \ldots, 6$ subject to the following nonzero Lie brackets:

$$\begin{aligned} [X_1, X_3] &= X_5 \qquad & [X_1, X_4] &= X_6 \\ [X_2, X_4] &= -X_5 \qquad & [X_2, X_3] &= X_6 \end{aligned} \tag{4.33}$$

and the left-invariant metric is induced from the inner product making the X_i a pseudo-orthonormal frame with X_6 timelike. Notice that this inner product is *not* ad-invariant:

$$1 = \langle [X_1, X_3], X_5 \rangle \neq \langle X_1, [X_3, X_5] \rangle = 0\,, \tag{4.34}$$

whence the metric on the group is not bi-invariant.

Let us introduce a dual basis θ^i for $\mathfrak{m}^*$ which we extend to left-invariant one-forms on the group M. They obey the Maurer-Cartan structure equation

$$d\theta^i(X, Y) = -\theta^i([X, Y])\,, \tag{4.35}$$

whence $d\theta^i = 0$ for $i = 1, 2, 3, 4$ and

$$d\theta^5 = -\theta^{13} + \theta^{24} \qquad \text{and} \qquad d\theta^6 = -\theta^{14} - \theta^{23}\,, \tag{4.36}$$

where we have used the notation $\theta^{ij} = \theta^i \wedge \theta^j$. To integrate these equations, we introduce coordinate functions x_i such that $\theta^i = dx_i$ for $i = 1, 2, 3, 4$ and

$$\theta^5 = dx_5 + x_3 dx_1 - x_4 dx_2 \qquad \text{and} \qquad \theta^6 = dx_6 + x_4 dx_1 + x_3 dx_2\,, \tag{4.37}$$

relative to which the metric is given by

$$\sum_{i=1}^{4} dx_i^2 + (dx_5 + x_3 dx_1 - x_4 dx_2)^2 - (dx_6 + x_4 dx_1 + x_3 dx_2)^2\,, \tag{4.38}$$

which exhibits M as an $\mathbb{R}^2$-bundle over flat $\mathbb{R}^4$, or as a real line bundle over a five-dimensional Gödel metric

$$\sum_{i=1}^{4} (dx^i)^2 - (dx_6 + x_4 dx_1 + x_3 dx_2)^2\,, \tag{4.39}$$

in different coordinates to the one in (4.16). In this sense it is to be compared with the maximally supersymmetric plane wave in six-dimensional $(1, 0)$ supergravity [32] which is also a line bundle over the Gödel metric (see, for example, [35]). Parenthetically, this prompts the natural question whether this six-dimensional g.o. geometry can support flux making it a homogeneous $(1, 0)$ supergravity background. The answer to this question is negative.

4.4.2 Isometries

The Lie algebra of isometries of M is a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$, where $\mathfrak{h}$ consists of those (outer) derivations of $\mathfrak{m}$ which are skew-symmetric relative to the inner product $\langle -, - \rangle$. (In the riemannian case this follows from a theorem of Gordon's [36].) Let δ be an outer derivation. Then δ preserves the centre $\mathfrak{z}$, which is the span of X_5, X_6. Since δ is skew-symmetric, it also preserves the orthogonal complement $\mathfrak{a} = \mathfrak{z}^\perp$ of the centre, spanned by X_i, $i = 1, 2, 3, 4$. The Lie bracket in $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{z}$ defines a map

$$\Lambda^2_+ \mathfrak{a} \to \mathfrak{z} \tag{4.40}$$

which is equivariant under the action of δ. It is not hard to show that δ must in fact act trivially in both spaces, which means that $\mathfrak{h} = \mathfrak{so}(\mathfrak{a})_- \subset \mathfrak{so}(\mathfrak{a})$ are anti-selfdual rotations in $\mathfrak{a}$, whence $\mathfrak{h} \cong \mathfrak{su}(2)$. Let Y_a, $a = 1, 2, 3$, denote a basis for $\mathfrak{h}$. Then the nonzero Lie brackets of $\mathfrak{g}$ are given by (4.33) together with

$$\begin{array}{lll}
[Y_1, X_1] = X_3 & [Y_2, X_1] = X_4 & [Y_3, X_1] = X_2 \\
[Y_1, X_2] = X_4 & [Y_2, X_2] = -X_3 & [Y_3, X_2] = -X_1 \\
[Y_1, X_3] = -X_1 & [Y_2, X_3] = X_2 & [Y_3, X_3] = -X_4 \\
[Y_1, X_4] = -X_2 & [Y_2, X_4] = -X_1 & [Y_3, X_4] = X_3 \\
[Y_1, Y_2] = -2Y_3 & [Y_2, Y_3] = -2Y_1 & [Y_3, Y_1] = -2Y_2 \,.
\end{array} \tag{4.41}$$

It is possible to write down the Killing vectors explicitly, in the coordinates above. First of all we notice that these coordinates are such that the Maurer-Cartan one-form $\boldsymbol{\theta} = \sum_{i=1}^6 \theta^i \otimes X_i$ is given by $\boldsymbol{\theta} = g(x)^{-1} dg(x)$, where

$$g(x) = \exp(x^1 X_1 + x^2 X_2) \exp(x^3 X_3 + x^4 X_4) \exp(x^5 X_5 + x^6 X_6) \,. \tag{4.42}$$

The Killing vectors in $\mathfrak{m}$ are the right-invariant vector fields on M, since they generate left-translations. Equivalently they are dual to the right-invariant one-form

$$\begin{aligned} dg(x) g(x)^{-1} &= \sum_{i=1}^4 dx^i \otimes X_i + (dx_5 + x_1 dx_3 - x_2 dx_4) \otimes X_5 + \\ &\quad + (dx_6 + x_1 dx_4 + x_2 dx_3) \otimes X_6 \,; \end{aligned} \tag{4.43}$$

that is,

$$\xi_{X_3} = \partial_3 - x_1 \partial_5 - x_2 \partial_6 \qquad \text{and} \qquad \xi_{X_4} = \partial_4 + x_2 \partial_5 - x_1 \partial_6 \tag{4.44}$$

and $\xi_{X_i} = \partial_i$ for $i = 1, 2, 5, 6$, where $\partial_i = \partial / \partial x^i$. Notice that, as expected,

$$[\xi_{X_i}, \xi_{X_j}] = -\xi_{[X_i, X_j]} \,. \tag{4.45}$$

The Killing vectors in $\mathfrak{h}$ are found by differentiating

$$\left. \frac{d}{dt} \right|_0 \mathrm{Ad}(\exp(tY)) g(x) \,, \tag{4.46}$$

for any $Y \in \mathfrak{h}$. Doing so we obtain

$$\begin{aligned}
\xi_{Y_1} &= -x_3\partial_1 - x_4\partial_2 + x_1\partial_3 + x_2\partial_4 - \frac{1}{2}(x_1^2 - x_2^2 - x_3^2 + x_4^2)\partial_5 - (x_1x_2 - x_3x_4)\partial_6 \\
\xi_{Y_2} &= -x_4\partial_1 + x_3\partial_2 - x_2\partial_3 + x_1\partial_4 - (x_1x_2 + x_3x_4)\partial_5 + \frac{1}{2}(x_1^2 - x_2^2 + x_3^2 - x_4^2)\partial_6 \\
\xi_{Y_3} &= -x_2\partial_1 + x_1\partial_2 + x_4\partial_3 - x_3\partial_4 \,.
\end{aligned} \tag{4.47}$$

Again it can be checked that as expected, the map $X \mapsto \xi_X$, for $X \in \mathfrak{g}$, is a Lie algebra anti-homomorphism: $[\xi_X, \xi_Y] = -\xi_{[X,Y]}$.

4.4.3 Null geodesics

It is easy to describe the null geodesics on M up to isometry. By homogeneity we can have them pass by any point, in particular the identity of M, thought of as a Lie group. The tangent vector is then a null vector $\boldsymbol{v} = \sum_{i=1}^6 v^i X_i \in \mathfrak{m}$, with $\sum_{i=1}^5 v_i^2 = v_6^2$. We can choose without loss of generality $v_6 = \pm 1$, depending on whether it is future- or past-pointing, respectively. The isotropy group $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ leaves X_5 invariant and acts transitively on the spheres in the orthogonal four-dimensional space. This means that up to isometry, there is a (quarter-)circle family of past- and future-pointing null geodesics, with tangent vectors

$$\boldsymbol{v} = \sin\vartheta X_1 + \cos\vartheta X_5 \pm X_6 \,, \tag{4.48}$$

for $\vartheta \in [0, \frac{\pi}{2}]$.

4.4.4 Geodesic orbits

The geodesic orbit nature of this homogeneous space is easy to see. Remember that this requires finding, for every $0 \neq X \in \mathfrak{m}$ a $\phi(X) \in \mathfrak{h}$ such that $X + \phi(X)$ is geodetic; that is, such that equation (2.39) is satisfied for all $Z \in \mathfrak{m}$. One finds that if $X = \sum_{i=1}^6 v_i X_i$, then letting $\phi(X) = \sum_{a=1}^3 \phi_a Y_a$, where

$$\begin{aligned}
\phi_1 &= (v_1^2 - v_2^2 + v_3^2 - v_4^2)\frac{v_5}{|\boldsymbol{v}_\perp|^2} - 2(v_1v_2 + v_3v_4)\frac{v_6}{|\boldsymbol{v}_\perp|^2} \\
\phi_2 &= -2(v_1v_2 - v_3v_4)\frac{v_5}{|\boldsymbol{v}_\perp|^2} - (v_1^2 - v_2^2 - v_3^2 + v_4^2)\frac{v_6}{|\boldsymbol{v}_\perp|^2} \\
\phi_3 &= 2(v_1v_4 + v_2v_3)\frac{v_5}{|\boldsymbol{v}_\perp|^2} + 2(v_1v_3 - v_2v_4)\frac{v_6}{|\boldsymbol{v}_\perp|^2} \,,
\end{aligned} \tag{4.49}$$

where $|\boldsymbol{v}_\perp|^2 = \sum_{i=1}^4 v_i^2$, yields a geodetic vector with $c = 0$ in equation (2.39). Notice that for the null geodesic with tangent vector $\boldsymbol{v}$ given by (4.48), we find

$$\phi_1 = v_5 = \cos\vartheta \qquad \phi_2 = -v_6 = \mp 1 \qquad \phi_3 = 0 \,, \tag{4.50}$$

which is the restriction of a linear function. In other words, whereas $\phi : \mathfrak{m}\backslash\{0\} \to \mathfrak{h}$ is nonlinear (showing that M is not naturally reductive), it is in some sense like a naturally reductive space when restricted to (certain) null geodesics.

4.4.5 Plane-wave limits

Let us consider the geodetic vector

$$\sin\vartheta X_1 + \cos\vartheta X_5 \pm X_6 + \cos\vartheta Y_1 \mp Y_2\,. \tag{4.51}$$

Using equation (3.53), we find that

$$F = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & \pm\frac{1}{2}\cos\vartheta \\ 0 & 0 & \mp\frac{3}{2} & -\frac{3}{2}\cos\vartheta \\ \frac{1}{2} & \pm\frac{3}{2} & 0 & 0 \\ \mp\frac{1}{2}\cos\vartheta & \frac{3}{2}\cos\vartheta & 0 & 0 \end{pmatrix}, \tag{4.52}$$

and, using (3.55), that

$$A = \begin{pmatrix} \frac{1}{8}(-3-\cos 2\vartheta) & \mp\frac{3}{4}\sin^2\vartheta & 0 & 0 \\ \mp\frac{3}{4}\sin^2\vartheta & \frac{1}{8}(-3-\cos 2\vartheta) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\cos 2\vartheta & 0 \\ 0 & 0 & 0 & \frac{1}{4}(-3+2\cos 2\vartheta) \end{pmatrix}. \tag{4.53}$$

It is easy to see that $[A, F] = 0$ if and only if $\vartheta = 0$, in which case the resulting spacetime is a conformally flat symmetric plane wave.

4.4.6 Homogeneous structures

We start with the reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{g}$ the full isometry algebra. The resulting homogeneous structure can be calculated using equation (2.32). Doing so we find a homogeneous structure of type $\mathcal{T}_2 \oplus \mathcal{T}_3$, with components $S_{ijk} = S(X_i, X_j, X_k)$ given by

$$\begin{aligned} S_{135} = S_{326} = S_{416} = S_{425} = S_{524} = S_{614} = S_{623} &= \frac{1}{2} \\ S_{146} = S_{236} = S_{245} = S_{315} = S_{513} &= -\frac{1}{2}\,. \end{aligned} \tag{4.54}$$

As explained at the end of section 2.2, we can search for other homogeneous structures by restricting to subalgebras $\mathfrak{g}' \subseteq \mathfrak{g}$ and looking for reductive splits $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$, where $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ and $\mathfrak{m}'$ is the graph of an $\mathfrak{h}'$-equivariant linear map $\mathfrak{m} \to \mathfrak{h}'$.

First of all we notice that there are no nontrivial $\mathfrak{h}$-equivariant linear maps $\mathfrak{m} \to \mathfrak{h}$, since decomposing $\mathfrak{h}$ and $\mathfrak{m}$ into irreducible $\mathfrak{h}$-modules, we see that they have no isotypical submodules in common: $\mathfrak{h}$ is simple, whence irreducible and three-dimensional, whereas $\mathfrak{m}$ breaks up into two one-dimensional trivial submodules and an irreducible four-dimensional submodule. Therefore to obtain other homogeneous structures, we must consider proper subalgebras $\mathfrak{g}' \subsetneq \mathfrak{g}$. It is only necessary to consider subalgebras up to conjugation, whence there is only one possibility: any one-dimensional subalgebra $\mathfrak{h}' \subset \mathfrak{h}$, e.g., the one spanned by Y_1, say. Any other choice is related by conjugation and will give rise to isomorphic homogeneous structures.

Decomposing $\mathfrak{m}$ and $\mathfrak{h}'$ into irreducible representations of $\mathfrak{h}'$ we find

$$\mathfrak{m} = \mathbb{R}_0 \oplus \mathbb{R}_0 \oplus \mathbb{R}_1^2 \oplus \mathbb{R}_1^2 \qquad \text{and} \qquad \mathfrak{h}' = \mathbb{R}_0\,, \tag{4.55}$$

where the subscripts indicate the highest weight of the representation. The trivial representations in $\mathfrak{m}$ are spanned by X_5 and X_6, respectively, whereas the two-dimensional representations are spanned by X_1, X_3 and X_2, X_4, respectively. We therefore have a two-parameter family of $\mathfrak{h}'$-equivariant linear maps $\varphi : \mathfrak{m} \to \mathfrak{h}'$, given by

$$\varphi(v^i X_i) = (\alpha v^5 + \beta v^6) Y_1 \,. \tag{4.56}$$

The graph of φ is then the subspace $\mathfrak{m}' \subset \mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}$ spanned by

$$X_1\,, \quad X_2\,, \quad X_3\,, \quad X_4\,, \quad X_5 + \alpha Y_1\,, \quad \text{and} \quad X_6 + \beta Y_1\,. \tag{4.57}$$

This means that the $[-,-]_{\mathfrak{m}'}$ brackets change; for example,

$$\begin{aligned}
[X_5 + \alpha Y_1, X_1]_{\mathfrak{m}'} &= \alpha X_3 & [X_6 + \beta Y_1, X_1]_{\mathfrak{m}'} &= \beta X_3 \\
[X_5 + \alpha Y_1, X_2]_{\mathfrak{m}'} &= \alpha X_4 & [X_6 + \beta Y_1, X_2]_{\mathfrak{m}'} &= \beta X_4 \\
[X_5 + \alpha Y_1, X_3]_{\mathfrak{m}'} &= -\alpha X_1 & [X_6 + \beta Y_1, X_3]_{\mathfrak{m}'} &= -\beta X_1 \\
[X_5 + \alpha Y_1, X_4]_{\mathfrak{m}'} &= -\alpha X_2 & [X_6 + \beta Y_1, X_4]_{\mathfrak{m}'} &= -\beta X_2 \,.
\end{aligned} \tag{4.58}$$

We can now compute the corresponding homogeneous structure using formula (2.32) and we obtain a two-parameter family of $\mathcal{T}_2 \oplus \mathcal{T}_3$ structures:

$$\begin{aligned}
S_{326} = S_{416} = S_{614} = S_{623} &= \frac{1}{2} & S_{146} = S_{246} &= -\frac{1}{2} \\
S_{316} = S_{426} = S_{613} = S_{624} &= \tfrac{1}{2}\beta & S_{136} = S_{246} &= -\frac{1}{2}\beta \\
S_{135} &= \frac{1}{2}(1+\alpha) & S_{245} &= -\frac{1}{2}(1-\alpha) \\
S_{315} = S_{513} &= -\frac{1}{2}(1+\alpha) & S_{425} = S_{524} &= \frac{1}{2}(1-\alpha) \,.
\end{aligned} \tag{4.59}$$

Naturally, when $\alpha = \beta = 0$ we recover the homogeneous structure of the maximal reductive split.

It is instructive to compare this with the explicit solution of the Ambrose-Singer equations (2.9). Solving these equations gives a general solution labelled by six parameters $z_1, \ldots, z_6$ in the intersection of three quadrics:

$$z_1 z_5 = z_2 z_4 \qquad z_1 z_6 = z_3 z_4 \qquad z_3 z_5 = z_2 z_6 \,. \tag{4.60}$$

These equations are equivalent to the matrix

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix} \tag{4.61}$$

having rank < 2. The general solution of such equations is given in terms of two vectors $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$ and $\boldsymbol{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, by

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \end{pmatrix} . \tag{4.62}$$

It is not hard to show that two such homogeneous structures labelled by $(\boldsymbol{v}, \boldsymbol{w})$ and $(\boldsymbol{v}', \boldsymbol{w}')$ are isomorphic if and only if $\boldsymbol{w}$ and $\boldsymbol{w}'$ are related by an $\mathrm{SO}(3) = \mathrm{Ad}\,\mathrm{SU}(2)$ transformation. Any $\boldsymbol{w} \in \mathbb{R}^3$ is SO(3)-related to $(w_1, 0, 0)$, in which case the solution has only two parameters $v_1 w_1$ and $v_2 w_1$, corresponding to our α and β above.

4.5 Komrakov spacetimes

In [10] there is a classification of four-dimensional pseudo-riemannian (locally) homogeneous spaces. The Komrakov list is a useful source of examples on which to test conjectures. In this section we will present two of them to illustrate the discussion in the bulk of the paper. The nomenclature follows [10].

4.5.1 Komrakov $1.1^2_{\lambda=0}$

This case in Komrakov's classification has a parameter (λ) which we are putting to zero in order for the resulting homogeneous space to admit a lorentzian metric. For $\lambda \neq 0$ the metric is either riemannian or hyperbolic. The isometry algebra is a semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ with $\mathfrak{h}$ one-dimensional with basis e_1 and $\mathfrak{m}$ four-dimensional with basis $u_1, \ldots, u_4$. The nonzero Lie brackets are

$$\begin{aligned} [u_4, u_1] &= -u_1 & [u_4, u_3] &= -u_3 \\ [u_4, u_2] &= -2u_2 & [u_1, u_3] &= -u_2 \\ [e_1, u_1] &= u_3 & [e_1, u_3] &= -u_1\,. \end{aligned} \tag{4.63}$$

Up to homothety (and Lie algebra automorphisms), there is a two-parameter family of $\mathfrak{h}$-invariant lorentzian metrics $\langle u_1, u_1 \rangle = \langle u_3, u_3 \rangle = 1$, $\langle u_2, u_2 \rangle = \alpha$ and $\langle u_4, u_4 \rangle = \beta$, with $\alpha\beta < 0$.

The homogeneous structure corresponding to this split is given by equation (2.32) and has (nonzero) components $S_{ijk} = S(u_i, u_j, u_k)$ given by

$$S_{123} = S_{213} = S_{312} = \frac{1}{2}\alpha \qquad S_{224} = -2\alpha \qquad S_{114} = S_{334} = -1\,, \tag{4.64}$$

which is of generic type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

It is possible to deform this homogeneous structure by choosing a different reductive split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}'$ with $\mathfrak{m}'$ the graph of an $\mathfrak{h}$-equivariant linear map $\varphi : \mathfrak{m} \to \mathfrak{h}$. We find that there is a 2-parameter family of such maps, and hence a 2-parameter family of such splits. Indeed, let $\mathfrak{m}'$ denote the span of the following vectors

$$u_1\,, \quad u_2 + c_2 e_1\,, \quad u_3\,, \quad \text{and} \quad u_4 + c_4 e_1\,, \tag{4.65}$$

with resulting homogeneous structure

$$\begin{aligned} &S_{123} = S_{312} = \frac{1}{2}\alpha \qquad S_{213} = c_2 + \frac{1}{2}\alpha \qquad S_{224} = -2\alpha \\ &S_{114} = S_{334} = -1 \qquad S_{413} = c_4\,. \end{aligned} \tag{4.66}$$

For generic values of c_2, c_4 this is again of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$, but there is a point, $c_2 = \frac{1}{2}\alpha$ and $c_4 = 0$, for which the $\mathcal{T}_3$ component is absent.

Up to the action of the isotropy, a null vector (at the identity coset) can be written as

$$v^1 u_1 + v^2 u_2 + v^4 u_4\,, \tag{4.67}$$

where $(v^1)^2 + \alpha(v^2)^2 + \beta(v^4)^2 = 0$. We must distinguish between two cases: $\alpha < 0, \beta > 0$ and $\alpha > 0, \beta < 0$. In either case, the timelike component can be set to 1 (for future-pointing null rays) without loss of generality.

- $\alpha < 0, \beta > 0$.

  In this case, the null vector is $u_2 + pu_4 + qu_1$, with $q = \sqrt{-\alpha - \beta p^2}$. We find that the geodetic equation (2.38) has a unique solution, with geodetic vector

$$u_2 + pu_4 \qquad \text{with } p^2 = -\alpha/\beta \text{ and } c = -2p. \tag{4.68}$$

  The plane-wave limit along this homogeneous geodesic will give rise to a singular homogeneous plane wave.

- $\alpha > 0, \beta < 0$.

  In this case, the null vector is $u_4 + pu_2 + qu_1$, with $q = \sqrt{-\beta - \alpha p^2}$. Here we find two homogeneous geodesics:

$$\begin{aligned} &u_4 + pu_2 \qquad \text{with } p^2 = -\beta/\alpha \text{ and } c = -2, \\ &u_4 + qu_1 \qquad \text{with } q^2 = -\beta \text{ and } c = -1. \end{aligned} \tag{4.69}$$

  Again the corresponding plane-wave limits will give rise to singular homogeneous plane waves.

For ease of exposition we will take $|\alpha| = |\beta| = 1$ from now on. We consider three cases:

- $v = u_2 \pm u_4$, $c = \mp 2$, $\alpha = -1$, $\beta = 1$.

  In this case, the skew-symmetric matrix F has components $F_{12} = \frac{1}{2}$, whereas the symmetric matrix A is given by

$$A = \begin{pmatrix} \frac{3}{4} & \pm 1 \\ \pm 1 & \frac{3}{4} \end{pmatrix}. \tag{4.70}$$

  It is clear that $[F, A] \neq 0$. Indeed,

$$e^{zF} A e^{-zF} = \begin{pmatrix} \frac{3}{4} \pm \sin z & \pm \cos z \\ \pm \cos z & \frac{3}{4} \mp \sin z \end{pmatrix}. \tag{4.71}$$

- $v = u_4 \pm u_2$, $c = -2$, $\alpha = 1$, $\beta = -1$. In this case, the skew-symmetric matrix F has components $F_{12} = \mp\frac{1}{2}$, whereas the symmetric matrix A is given by

$$A = \begin{pmatrix} \frac{3}{4} & \mp 1 \\ \mp 1 & \frac{3}{4} \end{pmatrix}. \tag{4.72}$$

  Again, it is clear that $[F, A] \neq 0$.

- $v = u_4 \pm u_1$, $c = 1$, $\alpha = 1$, $\beta = -1$.

  Finally, in this case, the skew-symmetric matrix F has components $F_{12} = \mp\frac{1}{2}$, whereas the symmetric matrix A is given by

$$A = \begin{pmatrix} \frac{15}{4} & \pm 1 \\ \pm 1 & \frac{7}{4} \end{pmatrix}. \tag{4.73}$$

  Again $[A, F] \neq 0$ and indeed

$$e^{zF} A e^{-zF} = \begin{pmatrix} \frac{11}{4} + \cos z - \sin z & \pm(\cos z + \sin z) \\ \pm(\cos z + \sin z) & \frac{11}{4} - \cos z + \sin z \end{pmatrix}. \tag{4.74}$$

4.5.2 Komrakov 1.4.6

The isometry algebra is the semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}$ of a one-dimensional Lie algebra $\mathfrak{h}$ spanned by e_1 and a four-dimensional Lie algebra spanned by $u_1, \ldots, u_4$. The nonzero Lie brackets are

$$\begin{aligned} [u_1, u_4] &= u_1 \\ [u_2, u_4] &= u_2 \\ [u_3, u_4] &= u_1 + u_3 \end{aligned} \qquad \begin{aligned} [e_1, u_2] &= u_1 \\ [e_1, u_3] &= u_2 \,. \end{aligned} \tag{4.75}$$

Up to homothety (and Lie algebra automorphism) there is a unique $\mathfrak{h}$-invariant lorentzian inner product on $\mathfrak{m}$: $\langle u_1, u_3 \rangle = -1$ and $\langle u_2, u_2 \rangle = \langle u_4, u_4 \rangle = 1$.

There is a two-parameter family of $\mathfrak{h}$-equivariant linear map $\mathfrak{m} \to \mathfrak{h}$. The graph $\mathfrak{m}'$ of a map in this family (labelled by α and β) is the subspace of $\mathfrak{g}$ spanned by

$$u_1 + \alpha e_1\,, \quad u_2\,, \quad u_3\,, \quad \text{and} \quad u_4 + \beta e_1\,. \tag{4.76}$$

The subspace $\mathfrak{m}'$ is no longer a Lie subalgebra, but projecting the brackets to $\mathfrak{m}'$ we obtain

$$\begin{aligned} [u_1 + \alpha e_1, u_4 + \beta e_1]_{\mathfrak{m}'} &= u_1 + \alpha e_1 \\ [u_1 + \alpha e_1, u_3]_{\mathfrak{m}'} &= \alpha u_2 \\ [u_1 + \alpha e_1, u_2]_{\mathfrak{m}'} &= \alpha(u_1 + \alpha e_1) \end{aligned} \qquad \begin{aligned} [u_2, u_4 + \beta e_1]_{\mathfrak{m}'} &= u_2 - \beta(u_1 + \alpha e_1) \\ [u_3, u_4 + \beta e_1]_{\mathfrak{m}'} &= u_1 + u_3 - \beta u_2 \,. \end{aligned} \tag{4.77}$$

The resulting homogeneous structure has components $S_{ijk} = S(u_i, u_j, u_k)$ given by

$$S_{134} = S_{314} = S_{334} = 1 \qquad S_{123} = -\alpha \qquad S_{423} = -\beta \qquad S_{224} = -1\,, \tag{4.78}$$

which is generically of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$, but of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ when $\alpha = \beta = 0$.

To determine the plane-wave limits along homogeneous geodesics, we first determine the null directions up to the action of isometries. Let $v = \sum_i v^i u_i \in \mathfrak{m}$ be a null vector. Then

$$2v^1 v^3 = (v^2)^2 + (v^4)^2\,. \tag{4.79}$$

The action of the isotropy is obtained by exponentiating the adjoint action of $e_1 \in \mathfrak{h}$:

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix} = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix} = \begin{pmatrix} v^1 + tv^2 + \frac{1}{2}t^2 v^3 \\ v^2 + tv^3 \\ v^3 \\ v^4 \end{pmatrix}. \tag{4.80}$$

We must distinguish between two cases:

- If $v^3 = 0$ then so are v^2 and v^4, whereas $v^1 \neq 0$. Therefore the null vector can be chosen to be u_1.
- If $v^3 \neq 0$ then we can use the isotropy action to put $v^2 = 0$ and rescale the null vector to make $v^3 = 1$, so that the null vector is then $u_3 + \alpha u_4 + \frac{1}{2}\alpha^2 u_1$ for some $\alpha \in \mathbb{R}$.

A simple calculation shows that in the first case, u_1 satisfies equation (2.38) with $c = 0$ when the plane-wave limit along that geodesic will be regular. In contrast, for the second case, there is no value of α for which the corresponding geodesic is homogeneous.

It is now a simple exercise to use equations (3.53) and (3.55) to show that the plane-wave limit along u_1 is actually flat.

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